ZDENĚK FROLÍK
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RNDr. Zdeněk Frolík, DrSc., died on May 3, 1989. Czechoslovak mathematical community thus has lost not only one of its greatest scientific personalities but also a tireless organizer of conferences and seminars, a devoted teacher whom the younger generation cannot forget, and a man who put honesty in his work and the level of Czechoslovak mathematics above all.

Zdeněk Frolík was born on March 10, 1933 in Zlonice. After completing Jungmann secondary school in the town of Litoměřice he studied mathematics at the Faculty of Mathematics and Physics, Charles University in Prague (1952–1957). Then he continued as a predoctoral student (aspirant) of Prof. Miroslav Katětov, defending his Ph. D. (Candidate of Science) thesis in 1959. Till 1964 he published 34 papers and in 1964 received his Doctor of Science degree. He worked at the Faculty till 1965, when he joined the Mathematical Institute of the Czechoslovak Academy of Sciences. Here he was head of Department of Fundamental Mathematical Structures since 1976.

Since 1961 Z. Frolík led a number of seminars: in measure theory, functional analysis, general topology, mathematical structures, uniform spaces. He organized the Prague Topological Symposia and the Winter Schools in Abstract Analysis, now world-famous scientific events.

His death reached him unexpectedly, in the middle of intensive work. A number of his projects will remain unaccomplished, as for example his book on the theory of uniform spaces and their applications.

The next pages are intended to recall some of the most important Frolík’s mathematical results. Naturally enough, our choice is incomplete and has a personal tinge.

The first results of young Frolík concerned the covering properties of topological spaces. The motivation came from the descriptive set theory, which will be mentioned later. The fundamental object was the topological Tichonov space equipped with a countable system of coverings, on which Frolík studied the properties of the type of completeness. Within the years 1957—1963 he penetrated in a remarkable way in the core of the problems which are described by this situation. The best way how to describe his results is to illustrate them on the example of Čech complete spaces, that is Tichonov spaces that are $G_δ$-sets in some of their compactifications. Let us present several main theorems.

A space $X$ is paracompact Čech complete if and only if it is the preimage of a complete metric space in a perfect continuous mapping.

A countable product of paracompact Čech complete spaces is a paracompact Čech complete space.

The product of a metrizable and a paracompact Čech complete space is paracompact.

The crucial role in the proof of the last two results is played by the following Frolík’s assertion: The product of a continuous and a perfect continuous mapping is perfect. This theorem can be also viewed as a generalization of the classical Tichonov theorem on the product of compact spaces.

An open continuous image of a Čech complete space is Čech complete.

A preimage of a Čech complete space in a perfect continuous mapping is Čech complete.

There are two more results from the above mentioned period that cannot be omitted: first, the assertion that every discrete or separable metrizable space can be embedded as a closed subspace into the product of two countably compact spaces, and second, Frolík’s proof of Glickberg’s theorem: $βΠX_i = ΠβX_i$ if and only if all subproducts of the spaces $X_i$ are pseudocompact. Frolík’s proof has replaced the original Glickberg’s one in the monographs as it is more elegant, simpler, and allows for numerous generalizations.

In the years 1965—1967 Frolík several times visited the United States, where he made friends with W. Rudin and his wife M. E. Rudin, and intensively worked in the theory of ultrafilters. His results in this field have become classical.

698
It was already in Prague that Frolík together with Miroslav Katětov studied a relation on types of ultrafilters on a countable set, now called the Rudin-Frolík ordering. Frolík’s assertion on the cardinality of cuts in this ordering was of fundamental importance: For every type \( \tau \) of an ultrafilter on \( N \) the cardinality of all types less than \( \tau \) is at most continuum, while the cardinality of all types greater than \( \tau \) is \( 2^c \). Using this result Frolík then proved, without any additional axioms on the set theory, that the space of all uniform ultrafilters on a countable discrete set, that is \( \beta \omega - \omega \) is not homogeneous. Let us note that the same result, however, with the assumption of the Continuum Hypothesis, had been proved by W. Rudin in 1956. Now we know that the Continuum Hypothesis cannot be avoided in Rudin’s proof, and that the more general result required an essentially new idea.

In a very short time Frolík realized that his technique enabled him to prove more; his series of papers on nonhomogeneity eventually culminated with his theorem on nonhomogeneity: No infinite compact \( F \)-space is homogeneous (an important step was done by Kunen’s proof of existence of two noncomparable ultrafilters).

Naturally, when dealing with the nonhomogeneity of compact \( F \)-spaces, Frolík necessarily reflected on the properties of their continuous mappings into themselves. Also in this case his publications started with the study of one typical example, namely the space \( \beta N \), and culminated with the second famous theorem from this period of Frolík’s creative work: Every one-to-one continuous mapping of an extremally disconnected compact space into itself has a clopen (i.e. simultaneously closed and open) set of fixed points. This theorem is a consequence of Frolík’s result asserting that for every one-to-one continuous mapping \( f \) on an extremally disconnected compact space \( X \) into itself there exists a decomposition of the space \( X \) into four clopen sets \( X_0, X_1, X_2, X_3 \) such that \( f[X_0] = X_0 \) and \( f[X_i] \cap X_i = \emptyset \) for \( i = 1, 2, 3 \).

Frolík’s fixed point theorem is again the best possible. Frolík himself showed that neither the requirement of the mapping \( f \) being one-to-one nor the requirement of \( X \) being compact can be weakened. Ten years later J. van Mill gave an example of an autohomeomorphism of the space \( \beta N \setminus N \) which has a nowhere dense set of fixed points, thus demonstrating that in Frolík’s theorem the extremal disconnectedness cannot be replaced by the property of being an \( F \)-space.

As we have mentioned above, the descriptive set theory was Frolík’s life-interest. As will be seen from some examples, also here he was attracted by its relations with the other branches of mathematics.

Zdeněk Frolík was one of the founders of modern descriptive theory of sets and spaces in the fifties and sixties.

The classical theory, usually connected with the names of Suslin and Luzin, could not go beyond the framework of separable metric spaces. The key problem was to find appropriate concepts and to prove their viability; the fundamental notion was that of the analytic space (more precisely, from the historical point of view, we should call it the \( K \)-analytic space). There existed definitions introduced by G. Choquet
and M. Sion, nonetheless, Frolik suggested other definitions (by the way, as was shown much later by Jayne, all these definitions coincide even for Hausdorff spaces). Frolik's approach turned out to be of considerable impact, it facilitated the development of an elegant theory and became the origin for many further works in the field.

In order to define the analytic space Frolik made use of a parametrization: A space $X$ is called analytic if there is an upper semicontinuous compact-valued (usco) correspondence $F: N^N \rightarrow_{onto} X$. (Note that the mapping from the classical definition of the analytic set is replaced by a multivalued mapping.)

For an equivalent description he used the notion of complete covering (a countable system of coverings is complete if every filter which is Cauchy with respect to this system has an accumulation point): A space $X$ is analytic if and only if there is a complete sequence of countable coverings of the space $X$.

If we assume that the multivalued mapping in the definition of the analytic space has disjoint values for distinct points (i.e. a generalization of a one-to-one mapping) we obtain the definition of the Luzin space. Later Frolik proved the important fact that Luzin subspaces of a completely regular space are finitely additive (obviously Luzin spaces are $\sigma$-disjoint additive and not $\sigma$-additive).

As an illustration of the first separation principle let us introduce its consequence:

Let $P$ be an analytic space. Then $X \subset P$ is a Baire set if and only if both $X$ and $P - X$ are analytic. (Note that the Borel sets from the classical theorem are replaced by Baire sets.) Another elegant and deep result is formulated in the following theorem:

Let $A$ be an analytic space, $M$ a metrizable space, and let the mapping $f: A \rightarrow M$ be Baire measurable. Then $\text{Gr}(f)$ and $f[A]$ are analytic (hence $f[A]$ is separable) and $f: A \rightarrow f(A)$ is a measurable quotient (hence $Z \subset f[A]$ is a Baire set if and only if $f^{-1}[Z]$ is a Baire set). (Note that this implies, for instance, that a completely Baire-additive disjoint system in an analytic space must be countable.)

Z. Frolik also investigated the Closed Graph Theorem. The result was the theorem on Suslin graph:

Let $E$ be a topological linear space which is inductively generated by linear homomorphisms from topological spaces that are not of the 1st category in themselves, and let $F$ be a locally convex space whose topology is analytic (these assumptions are satisfied for Banach spaces). If $f: E \rightarrow F$ is a linear homomorphism whose graph is a Suslin set in $E \times F$, then $f$ is continuous. (Recall that Suslin sets in a given space are the sets resulting by the Suslin operation from closed sets, and that analytic subspaces are Suslin spaces.)

Z. Frolik devoted much effort to the development of nonseparable descriptive theory (more precisely, the descriptive theory in topological spaces that are not Lindelöf spaces). Here Hansell's lemma provided an important technical tool: A completely Suslin-additive disjoint system in a complete metric space is $\sigma$-discretely decomposable (a system $\mathcal{A}$ of subsets of a set $X$ is $\sigma$-discretely decomposable if
every $A \in \mathcal{A}$ can be expressed as $A = \bigcup_{n \in \omega} A_n$ so that the system \{\{A_n; A \in \mathcal{A}\}$ is discrete in $X$ for every $n \in \omega$.

In cooperation with P. Holický, Z. Frolík developed a theory based on the discreteness defined in terms of uniformity, mainly the fine uniformity. They defined that a space $X$ is analytic (more precisely, $\lambda$-analytic) if and only if there exists an uco correspondence $F: \lambda^\alpha \rightarrow_{onto} X$ preserving the $\sigma$-discretely decomposable systems (here $\lambda$ denotes both the cardinality and the discrete topological space of this cardinality).

They proved theorems analogous to the classical ones, and also the following result: $X$ is $\lambda$-analytic for some $\lambda$ if and only if $X$ is paracompact and there are $G \subseteq \beta X$, $A \subseteq \beta X$ such that $X = A \cap G$, $G$ is $G_\delta$ in $\beta X$ and $A$ is Suslin in $\beta X$.

We may say that Z. Frolík and P. Holický developed the theory of paracompact analytic sets (recall that analytic sets in the separable theory are Lindelöf and that the just presented characterization complies with the scheme suggested by Frolík for the construction of nonseparable concepts before Hansell's lemma was discovered).

It is evident that in the parametric definition of analyticity it is necessary to preserve some kind of discreteness. Using different types of discreteness (generating however identical $\sigma$-discretely decomposable systems in metric spaces), numerous authors (among them also Z. Frolík) introduced and studied various concepts of analytic sets. The works of R. Hansell, J. Jayne and C. A. Rogers suggest that the development of the theory and its applications is not yet concluded.

Finally, let us mention Čech analytic spaces. This notion was introduced by D. H. Fremlin: A space $X$ is Čech analytic if it results by the Suslin operation applied to Borel sets in $\beta X$. Frolík studied the problem whether the Čech analytic spaces are preserved by perfect mappings. He proved that the answer is affirmative if the range is metrizable, and conjectured that generally the answer is negative (which has been neither proved nor disproved yet).

Although the above survey of mathematical results of Zdeněk Frolík is far from being complete, it perhaps gives a sufficient view of his stature. Nevertheless, we cannot neglect another one of his achievements: the seminar on uniform spaces.

In early seventies Z. Frolík collected a group of young people in a seminar on uniform spaces. Graduates and predoctoral students from various directions (combinatorics, topology, mathematical analysis) formed under his guidance a team that was working in a unique creative atmosphere till the year 1979 and produced results which M. D. Rice at Prague Topological Symposium in 1981 appreciated in the following way: "You have got ahead of the rest of the world by at least ten years."

When founding the seminar, Frolík himself had in mind mainly the applications of uniform spaces in the measure theory and the descriptive theory of sets and spaces; his works and those of his students contain a fine and general theory of uniform measures. However, soon he realized that the seminar was able to cover a much wider domain. Frolík supported its progress with enthusiasm, inspiring the participants with ever new motives and suggestions. Theory of categorial refinements,
theory of uniform atom and theory of combinatorial complexity of uniform coverings — these are only some of the fruits of the seminar.

The Fate niggardly assigned to Zdeněk Frolík merely fifty six years of life. The memory of his will accompany his numerous students, and his achievements bind them to continue his work.

LIST OF PUBLICATIONS OF RNDr. ZDENĚK FROLÍK, DrSc.


703


[82] Basic refinements of the category of uniform spaces. TOPO 72 — General Topology and

[95] (with F. Zitek) On the occasion of the 70th birthday of Academician Josef Novák (Czech), Čas. pěst. mat., 100 (1975), 208—214.

[111] Recent development of theory of uniform spaces. General topology and its relations to


[121] (with J. Fried) A characterization of uniform paracompactness.


[150] (with I. Netuka) Čech completeness and fine topologies in potential theory and real analysis. (to appear in Expositio Math.).
[153] (with D. Dikranjan and E. Giuli) $A$ — closed spaces. (preprint)
[154] Generalization of some properties of complete metric spaces. (Czech).