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ON COUNTABLE VON NEUMANN REGULAR RINGS

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1. INTRODUCTION

In the present note, we study properties of countable von Neumann regular rings. Finite regular rings have simple homological properties since they are completely reducible (i.e., finite direct sums of full matrix rings over division rings). The same is far from being true of the countable ones. There are various examples of non-completely reducible countable regular rings, usually constructed as endomorphism rings of infinite-dimensional linear spaces or direct limits of completely reducible rings (see [3]). Homological properties of such rings may be independent of ZFC. For example, the Whitehead property of all (or some) non-zero countable modules over any simple countable non-completely reducible regular ring is independent of ZFC + GCH (see [7]). In fact, this property is assured by Jensen’s diamond, but excluded by a combinatorial principle due to Shelah.

In our note we show that a version of Shelah’s principle is even equivalent to a property concerning the bifunctor Ext that is close to the Whitehead property. We also get a structure theorem for countable regular \(\otimes\)-rings. Finally, we obtain a splitting-type theorem for modules over regular rings such that each left ideal is countably generated.

2. PRELIMINARIES

In what follows, an ordinal is identified with the set of its predecessors and a cardinal is an ordinal which is not equipotent with any of its predecessors. If \(\kappa\) is a cardinal, then \(\text{cf}(\kappa)\) denotes its cofinality and \(\kappa^+\) denotes the successor cardinal to \(\kappa\). For a set \(A\), the cardinality of \(A\) is denoted by \(\text{card}(A)\). Let \(E\) be a subset of \(\chi_1\). Then \(E\) is cofinal if \(\sup E = \chi_1\). Further, \(E\) is closed if \(\sup F \in E \cup \{\chi_1\}\) for every non-empty subset \(F\) of \(E\). We say that \(E\) is stationary if \(\chi_1 \cap E \neq \emptyset\) for every closed and cofinal subset \(F\) of \(\chi_1\). We say that \(E\) is costationary if \(\chi_1 - E\) is stationary.

\(^1\) \(\chi_0\) denotes the cardinality of the set of all natural numbers, \(\chi_1\) denotes the successor cardinal to \(\chi_0\).
In what follows, all rings are associative with unit. If $R$ is a ring and $n > 0$ a natural number, then $M_n(R)$ denotes the full matrix ring of degree $n$ over $R$. The Jacobson radical of a ring $R$ is denoted by $\text{Rad}(R)$. The category of unitary left and right $R$-modules are denoted by $\text{mod}$ and $\text{mod-R}$, respectively. A unitary left $R$-module is simply called a module. A sum and a direct sum of submodules are denoted by $\bigoplus$ and $\bigoplus$, respectively. Let $M$ be a module. If $\kappa$ is an ordinal, $\kappa > 0$, then $M^{(\kappa)}$ and $M^\kappa$ denote the direct sum and the direct product, respectively, of $\kappa$ copies of $M$. If $x \in M$, then $\text{Ann}(x)$ denotes the left annihilator of $x$ in $R$. Further, $M$ is said to be properly $\kappa$-generated if $\kappa$ is the smallest cardinal such that there is a generating set of $M$ of cardinality $\kappa$. The $Z$-module of rational numbers is denoted by $\mathbb{Q}$.

A ring is said to be a $\otimes$-ring if there are only trivial orthogonal theories of the tensor product bifunctor (see [5, Introduction]), i.e. $M \otimes N \neq 0$ for each non-zero $M \in \text{mod-R}$ and $N \in \text{R-mod}$. Recall that by [2, Appendix A] a ring is a left $T$-ring if there are only trivial orthogonal theories of the $\text{Ext}$ bifunctor, i.e. $\text{Ext}(M, N) \neq 0$ for each non-projective module $M$ and each non-injective module $N$. Further concepts and notation can be found e.g. in [1].

3. HOMOLOGICAL PROPERTIES OF COUNTABLE REGULAR RINGS

3.1. Lemma. Let $R$ be a simple non-completely reducible regular ring, $M$ a singular module and $K = \text{End}(M)$. Assume that the right $K$-module $M$ is countably generated. Then there are orthogonal idempotents $e_i$, $i < \chi_0$ such that $M = \sum_{i < \chi_0} e_i M$.

Proof. Let $\{x_i \mid i < \chi_0\}$ be a generating set of the right $K$-module $M$. We construct the idempotents $e_i$, $i < \chi_0$ by induction. First, put $e_0 = 1 - e$, where $e$ is any non-trivial idempotent such that $e \in \text{Ann}(x_0)$. Assume orthogonal idempotents $e_0, \ldots, e_{n-1}$ such that $1 = f = e_0 + \ldots + e_{n-1}$ and $\sum_{i < n} x_i K \subseteq \sum_{i < n} e_i M$ have been constructed. If $x_n \in \sum_{i < n} e_i M$, let $e_n$ be any non-trivial idempotent of the ring $(1 - f)$. 

3.2. Lemma. Let $R$ be a regular ring. Then the following conditions are equivalent:

(i) $R$ is a $\otimes$-ring.
(ii) \( I + J \neq R \) for each maximal right ideal \( I \) and each maximal left ideal \( J \).

Proof. (i) implies (ii). Easy.

(iii) implies (i). Assume \( A \otimes B = 0 \) for some non-zero \( A \in \text{mod-}R \) and \( B \in \text{R-mod} \). Then, for each cardinal \( \kappa \), \( \text{Hom}_Z (A \otimes B, (Q/Z)^\kappa) = 0 \), whence \( \text{Hom}_Z (A, Q/Z)^{\kappa} = 0 \). Since \( A \) is a flat right \( R \)-module, \( \text{Hom}_Z (A, Q/Z) \) is injective and not a cogenerator. By [1, Proposition 18.15], there is a simple module \( W \) such that \( \text{Hom}_Z (W, (Q/Z)^{\kappa}) = 0 \) for all cardinals \( \kappa \). Hence \( A \otimes W = 0 \). Using the right-hand homomorphisms, we get similarly the existence of a simple right \( R \)-module \( V \) satisfying \( V \otimes W = 0 \). Now, let \( I \) be a maximal right ideal with \( V \cong R/I \) and \( J \) a maximal left ideal with \( W \cong R/J \). Using the commutative diagram of [1, 19.17], it is easy to see that the canonical inclusion \( V \to V \) is a \( Z \)-isomorphism, whence \( I + J = R \).

3.3. Lemma. Let \( R \) be a regular ring. Then

(i) if \( R \) is a \( \odot \)-ring, then \( R \) is simple;

(ii) if each maximal right ideal is countably generated and all simple modules are isomorphic, then \( R \) is a \( \odot \)-ring;

(iii) if \( R \) is a simple left and right \( T \)-ring, then \( R \) is a \( \odot \)-ring.

Proof. (i) By 3.2, every two-sided ideal is a superfluous submodule of \( R \) and hence is contained in \( \text{Rad}(R) = 0 \).

(ii) Let \( I \) and \( J \) be a maximal right and left ideal, respectively. By [3, Proposition 2.14], there are a cardinal \( \kappa \leq \chi_0 \) and orthogonal idempotents \( e_i, i < \kappa \) such that \( I = \sum_{i < \kappa} e_i R \). Put \( I' = \sum_{i \leq \kappa} Re_i \). Then \( I' \neq R \), whence \( \text{Hom}(R/I', R/J) \neq 0 \). Let \( r \in R \) be such that \( r \notin J \) and \( e_i r \in J \) for all \( i < \kappa \). Then even \( r \notin I + J \), and 3.2 applies.

(iii) By [6, Theorem II.3], [2, Proposition A.3.5] and (ii).

3.4. Theorem. Let \( R \) be a countable regular ring. Then \( R \) is a \( \odot \)-ring if and only if there are a natural number \( n > 0 \) and a division ring \( D \) such that \( R \cong M_n(D) \).

Proof. The sufficiency is easy. Assume \( R \) is a countable regular ring. If \( R \) is not simple, then 3.3 (i) shows \( R \) is not a \( \odot \)-ring. If \( R \) is simple and non-completely reducible, take a simple module \( M \) and put \( K = \text{End}(M) \). Then \( \dim_K(M) = \chi_0 \), and 3.1 yields the existence of orthogonal idempotents \( e_i, i < \chi_0 \) such that \( M = \sum_{i < \chi_0} e_i M \). Let \( I \) be a maximal right ideal containing all \( e_i, i < \chi_0 \), and \( J \) a maximal left ideal such that \( M \cong R/J \). Then \( I + J = R \) and, by 3.2, \( R \) is not a \( \odot \)-ring.

Hence, \( R \) is simple and completely reducible, q.e.d.

Let \( E \) be a subset of \( \chi_1 \) and \( F \) the set of limit ordinals of \( E \). Let \( \varphi_E = (n_v \mid v \in E) \) be a sequence of strictly increasing \( \chi_0 \)-sequences such that, for each \( v \in F \), \( \sup n_v(i) = v \), and, for each \( i < \chi_0 \) and \( v \in F \), there is a limit ordinal \( p_v(i) \) with \( n_v(i) = p_v(i) + i + 1 \). Denote by \( C_{E,F} \) the following combinatorial principle: „for any sequence \( (h_v \mid v \in E) \) of functions from \( \chi_0 \) to \( \chi_0 \) there is a function \( f: \chi_1 \to \chi_0 \) such that \( \forall v \in F \exists j_v < \chi_0 \forall i > j_v: (n_v(i)) f = (i) h_v \).”
Let $R$ be a simple countable non-completely reducible regular ring and let $\mathcal{S} = (I_v \mid v \in F)$ be a sequence of properly $\chi_0$-generated left ideals of $R$. By [3, Proposition 2.14], for each $v \in F$ there are orthogonal idempotents $e_{iv}, i < \chi_0$ such that $I_v = \sum_{i<\chi_0} R e_{iv}$. Denote by $M_{(E,\mathcal{S})}$ the module $R^{(\chi_1)}|G$, where $G$ is a submodule of $R^{(\chi_1)}$ generated by the elements $g_{iv} \in R^{(\chi_1)}, i < \chi_0, v \in F$, the $v$-th projection of $g_{iv}$ being $-e_{iv}$, the $n_i(i)$-th projection being $e_{iv}$, and all other projections being zero. Let $P$ be a simple module. Then the right dimension of $P$ over $\text{End}(P)$ is $\chi_0$, and by 3.1, there are orthogonal idempotents $e_i, i < \chi_0$ such that $P = \sum_{i<\chi_0} e_i P$. Denote by $I_p$ the left ideal of $R$ generated by the set $\{e_i \mid i < \chi_0\}$ and by $\mathcal{S}_p$ the constant sequence $(I_p \mid v \in F)$.

3.5. Theorem. Let $R$ be a simple countable non-completely reducible regular ring, let $E$ be a subset of $\chi_1$, and let $\mathcal{Q}_E$ be as above. Then the following conditions are equivalent:

(i) $C_{(E,\mathcal{Q}_E)}$;
(ii) $\text{Ext}(M_{(E,\mathcal{Q}_E)}, N) = 0$, for any countably generated module $N$ and any $\mathcal{S}$;
(iii) there is a simple module $P$ such that $\text{Ext}(M_{(E,\mathcal{Q}_E)}, P) = 0$.

Proof. (i) implies (ii). An easy generalization of [7, Theorem 2.2]. (ii) implies (iii). Obvious. (iii) implies (i). Let $(h_v \mid v \in E)$ be a sequence of functions from $\chi_0$ to $\chi_0$. Let $e_i, i < \chi_0$ be orthogonal idempotents such that $P = \sum_{i<\chi_0} e_i P$. Since $R$ is simple and countable, for each $i < \chi_0$ there is a bijection $r_i: e_i P \to \chi_0$. Define $p \in \text{Hom}(G, P)$ by $(g_{iv}) p = (i) h_v r_i^{-1}$ for $i < \chi_0$ and $v \in F$. Since $\text{Ext}(M_{(E,\mathcal{Q}_E)}, P) = 0$, there is a $q \in \text{Hom}(R^{(\chi_1)}, P)$ such that $(g_{iv}) q = (i) h_v r_i^{-1}$ for each $i < \chi_0$ and $v \in F$. Define a function $f: \chi_1 \to \chi_0$ by $\alpha f = (e_1 n_i(i) q) r_i$ if there are $i < \chi_0$ and $v \in F$ such that $\alpha = n_i(i)$, and $\alpha f = 0$ otherwise. Now, for each $v \in F$, there is a $j_v < \chi_0$ such that $1_v q = \sum_{i \leq j_v} e_i P$, whence $e_i 1_v q = 0$ for all $i > j_v$. Hence, for each $v \in F$ and each $i > j_v$, we get $(i) h_v = (g_{iv}) q r_i = (e_1 1_n(i) q - e_i 1_v q) r_i = (n_i(i)) f$, q.e.d.

3.6. Remark. Note that if $E$ is a stationary costationary subset of $\chi_1$, then, for any $\mathcal{Q}_E$, $C_{(E,\mathcal{Q}_E)}$ is independent of ZFC + GCH (see [4]). If $E$ is not stationary, then, for any simple module $P$ and any $\mathcal{Q}_E$, the module $M_{(E,\mathcal{Q}_E)}$ is projective, and hence $C_{(E,\mathcal{Q}_E)}$ holds for any $\mathcal{Q}_E$.

3.7. Lemma. Let $\kappa$ be an infinite cardinal and $R$ a ring such that each left ideal is $\kappa$-generated. Let $\lambda > 0$ be a cardinal, $F = R^{(\lambda)}$, and $I$ a submodule of $F$. Then $I$ is max ($\kappa$, $\lambda$)-generated.

Proof. We prove the assertion by induction on $\lambda$. It is clear for $\lambda = 1$. For $1 \leq \lambda < \chi_0$, let $I$ be a submodule of $R^{(\lambda+1)}$, $I = \sum_{x < \mu} R x_a$. In fact, $x_a = (y_a, z_a)$, where $y_a \in R^{(\lambda)}$ and $z_a \in R$ for each $x < \mu$. Hence, there is a set $A \subseteq \mu$ such that card ($A$) $\leq$
and, for each \( \alpha < \beta \), there is a finite subset \( A_\alpha \subseteq A \) and elements \( r_{\alpha \beta} \in R \), \( \beta \in A_\alpha \) such that \( z_\alpha = \sum_{\beta \in A_\alpha} r_{\alpha \beta} z_\beta \). Put \( B = \{(y_\alpha - \sum_{\beta \in A_\alpha} r_{\alpha \beta} y_\beta, 0) \mid \alpha \in (\mu - A)\} \). Then \( B \cup \{(y_\alpha, z_\alpha) \mid \alpha \in A\} \) is a generating set of \( I \). By the premise, \( B \) can be replaced by its subset of cardinality \( \leq \kappa \). For \( \lambda \) infinite (i.e. \( \lambda \) a limit ordinal), take a cofinal subset of ordinals \( \lambda_\alpha, \alpha < \text{cf}(\lambda) \). Then clearly \( I = \bigcup_{\alpha < \text{cf}(\lambda)} (I \cap R(\lambda_\alpha)) \) and the induction works.

Let \( R \) be a regular left hereditary ring, \( F \) a free module, and \( I \) a submodule of \( F \). Let \( \kappa \) be the cardinal such that \( I \) is properly \( \kappa \)-generated. By [1, Corollary 26.2] and [3, Proposition 2.14], there are \( x_\alpha \in I, \alpha < \kappa \) such that \( I = \sum_{\alpha < \kappa} R x_\alpha \). We can assume that \( F = R(\delta) \) for a cardinal \( \delta > 0 \). For \( \kappa < \delta \) denote by \( \xi_k \) the \( k \)-th natural projection of \( F \) to \( R \). For \( B \subseteq \kappa \) put \( J_B = \sum_{\alpha \in B} R x_\alpha \), and for \( C \subseteq \delta \) put \( F_C = \sum_{k \in C} F \xi_k \). If there is a finite set \( C \subseteq \delta \) such that \( I \subseteq F_C \), we say that \( I \) belongs to case 1. If there is a countable set \( C \subseteq \delta, C = \{c_i \mid i < \omega_0\} \), such that \( I \subseteq F_C \), but \( I \not\subseteq F_{C_n} \) for each \( n < \omega_0 \) and \( C_n = \{c_i \mid i \leq n\} \), we say that \( I \) belongs to case 2. Further, denote by \( \text{SPLIT}(I, F, \kappa) \) the following splitting property: „there is a subset \( A \subseteq \kappa \) such that \( \text{card}(A) = \kappa \) and \( J_A \cap F_{C_n} \) is finitely generated for all \( n < \omega_0 \). Moreover, if \( R \) is not completely reducible and (2) holds, then both possibilities (i.e. \( \text{SPLIT}(I, F, \omega_0) \), or \( \text{WSPLIT}(I, F, \omega_0) \)) but not \( \text{SPLIT}(I, F, \omega_0) \) can occur.

Proof. (i) Take a fixed free module \( F = R(\delta) \) and a properly \( \kappa \)-generated submodule \( I \) of \( F \). By 3.7 we have \( \kappa \leq \max(\kappa_0, \delta) \). If \( \kappa < \omega_0 \), then the assertion is well-known ([3, Theorem 1.11]). The rest of the proof of part (i) follows from the next two lemmas:

3.8. Theorem. Let \( R \) be a regular ring such that each left ideal is countably generated.

(i) If \( \kappa \neq \omega_0 \), then \( \text{SPLIT}(\kappa) \).

(ii) Let \( \kappa = \omega_0 \), let \( F \) be a free module and \( I \) a properly \( \omega_0 \)-generated submodule of \( F \). Then either

(1) \( I \) belongs to case 1 and not \( \text{WSPLIT}(I, F, \omega_0) \), or

(2) \( I \) belongs to case 2 and \( \text{WSPLIT}(I, F, \omega_0) \).

If (2) holds, then \( \text{SPLIT}(I, F, \omega_0) \) iff there is a subset \( A \subseteq \omega_0 \) such that \( \text{card}(A) = \omega_0 \) and \( J_A \cap F_{C_n} \) is finitely generated for all \( n < \omega_0 \). Moreover, if \( R \) is not completely reducible and (2) holds, then both possibilities (i.e. \( \text{SPLIT}(I, F, \omega_0) \), or \( \text{WSPLIT}(I, F, \omega_0) \)) but not \( \text{SPLIT}(I, F, \omega_0) \) can occur.

Proof. (i) Take a fixed free module \( F = R(\delta) \) and a properly \( \kappa \)-generated submodule \( I \) of \( F \). By 3.7 we have \( \kappa \leq \max(\kappa_0, \delta) \). If \( \kappa < \omega_0 \), then the assertion is well-known ([3, Theorem 1.11]). The rest of the proof of part (i) follows from the next two lemmas:

3.9. Lemma. Assume \( \omega_0 < \kappa \leq \delta \) and \( \text{cf}(\kappa) \neq \omega_0 \). Then \( \text{SPLIT}(\kappa) \).

Proof. We generalize the proof of [6, Theorem II.3] as follows. Let \( N \) be a module. Let \( P = \text{Hom}(I, N) \) and \( \lambda_0 = \text{card}(P) \), i.e. \( P = \{p_\gamma \mid \gamma < \lambda_0\} \). For \( i < \omega_0 \), put \( \lambda_{i+1} = \lambda_i^+ \) and let \( \lambda = \sup_{i < \omega_0} \lambda_i \). Clearly, \( \text{cf}(\lambda) = \omega_0 \). Let \( N_i, i < \omega_0 \), and \( N_\lambda \) be as
in [6, Lemma II.2]. Then \(N^2/N\) is injective, whence \(\text{Ext}(F/I, N^2/N) = 0\). Define \(f \in \text{Hom}(I, N^2/N)\) by \(x_f = n_x + N\), \(x < \kappa\), where \(n_x \pi_v = x vp_v\) if \(v < \lambda_0\), \(n_x \pi_v = x vp_v\) if \(v \in \lambda, \mu < \lambda_0\), and \(n_x \pi_v = 0\) otherwise. Then there exist \(y_k \in N^2, \ k < \delta\) such that \((\sum_j x_{\delta_j} y_k - n_a) \in N\) for each \(x < \kappa\). For \(i < \chi_0\), put \(A_i = \{x \in \kappa \mid \sum_k x_{\delta_k} y_k - n_a \in N\}\). Then \(A_i \subseteq A_{i+1}, \ i < \chi_0\), and \(\kappa = \bigcup A_i\), whence there is a \(j < \chi_0\) such that \(\text{card}(A_j) = \kappa\). Put \(A = A_j\). Then for each \(\lambda_j \leq \kappa\), \(\sum_k x_{\delta_k} y_k - n_a = 0\). Let \(g \in \text{Hom}(J_A, N)\). Then there is a \(\gamma < \lambda_0\) such that \(p_j/J_A = g\). Put \(\mu_0 = \lambda_j + \gamma\). Then for each \(x \in A, x\mu_0 = n_x \pi_{\mu_0} = (\sum_k x_{\delta_k} y_k) \pi_{\mu_0}\). Define \(h \in \text{Hom}(F, N)\) by \(1_k h = y_k \pi_{\mu_0}\), \(k < \delta\). Then \(x h = x h\) for each \(x \in A\). Now, consider the particular case of \(N = I\) and \(g\) the canonical inclusion of \(J_A\) to \(I\). Denote by \(g_1\) the canonical inclusion of \(J_A\) to \(F\) and by \(g_2\) the canonical projection of \(I\) to \(J_A\). Then \(1_J g = g_1 h g_2\), whence \(J_A = \text{im} g_1\) is a direct summand of \(F\).

3.10. Lemma. Assume \(\chi_0 < \kappa \leq \delta\) and \(\text{cf}(\kappa) = \chi_0\). Then \(\text{Split}(\kappa)\).

Proof. Let \(\kappa = \{i \in \kappa \mid \chi_0 < \kappa_i, \ k_i < \kappa_i+1, \ i < \chi_0\} \sup \kappa_i = \kappa\). By induction on \(i < \chi_0\) we construct sets of ordinals \(B_i \subseteq \kappa\) and \(C_i \subseteq \kappa\), \(i < \chi_0\) such that \(\kappa_i \in B_i \subseteq B_{i+1}, C_i \subseteq C_{i+1}\), \(\text{card}(B_i) = \kappa_i, J_{B_i} \subseteq F_{C_i}\), and \(J_{(\kappa - B)} \cap F_{C_i} = 0\). Put \(B_0 = C_0 = \kappa_0\) and assume that \(B_i\) and \(C_i\) are defined for some \(i < \chi_0\). Let \(D_0 = C_i \cup \{\kappa < \delta \mid \exists \alpha < \kappa_i+1; x_{\alpha} \pi_{\gamma} \neq 0\}\). By 3.7, \(\text{card}(D_0) = \kappa_i+1\). Assume \(D_j\) is defined for some \(j < \chi_0\) so that \(D_0 \subseteq D_j\) and \(\text{card}(D_j) = \kappa_j+1\). Let \(H\) be the set of finite subsets \(H \subseteq \kappa\) satisfying \(J_{K} \cap F_{D_j} = 0\) and \(J_{K} \cap F_{D_j} = 0\) for any proper subset \(H' \subseteq H\). Using 3.7 it is easy to see that \(\text{card}(H) = \kappa_i+1\). Put \(D_{j+1} = D_j \cup \{\kappa < \delta \mid \exists \alpha \in H; x_{\alpha} \pi_{\gamma} = 0\}\). Then \(\text{card}(D_{j+1}) = \kappa_i+1\). Now, it suffices to put \(B_{i+1} = \bigcup_{j < \chi_0} D_j\) and \(B_{i+1} = \{x \in \kappa \mid x \in F_{C_i+1}\}\). Further, for \(i < \chi_0\), put \(A_i = B_{i+1} - B_i\). If \(i > 0\) and \(a \in A_i\), then \(x_a = (y_a, z_a)\) for some \(y_a \in F_{C_i}\), \(z_a \in F_{(C_i+1)-C_i}\). Note that the elements \(z_a, \ a \in A_i\), are independent, as \(J_{(\kappa - B_i)} \cap F_{C_i} = 0\). If \(a \in A_0\), we put \(z_a = x_a\). Anyway, it follows from 3.9 that for each \(i < \chi_0\) there is a subset \(A' \subseteq A_i\) such that \(\text{card}(A') = \kappa_i+1\) and \(\sum_{z \in A'} Rz\) is a direct summand of \(F_{(C_i+1)-C_i}\). Now, it suffices to put \(A = \bigcup_{i < \chi_0} A_i\).

(ii) If \(I\) belongs to case 1, then obviously not \(\text{WSplit}(I, F, \chi_0)\). Assume \(I\) belongs to case 2. Then the assertion concerning \(\text{Split}(I, F, \chi_0)\) follows immediately from the fact that a submodule \(N\) of \(F_{C}\) is a direct summand if \(N \cap F_{C_n}\) is finitely generated for all \(n < \chi_0\) (see the proof of [6, Lemma III.3]). The rest of the proof of part (ii) follows from the following lemma:

3.11. Lemma. Let \(I\) belong to case 2. Then \(\text{WSplit}(I, F, \chi_0)\). If \(R\) is not complete-
ly reducible, there exist $I_1$ and $I_2$ such that $\text{SPLIT}(I_1, F, \chi_0)$, and $\text{WSPLIT}(I_2, F, \chi_0)$ but not $\text{SPLIT}(I_2, F, \chi_0)$.

Proof. Define a set \( \{ y_i \in I \mid i < \chi_0 \} \) as follows. First, \( y_0 = x_0 \). Assume \( y_i, i \leq n \) are independent for some \( n < \chi_0 \). Let \( j \) be the least natural number such that \( \sum_{i \leq n} R y_i \subseteq F_{c_j} \). Take \( i < \chi_0 \) such that \( x_i \notin F_{c_j} \). Then there are some \( u_i \in F_{c_j} \) and \( 0 \neq v_i \in F_{(C-C_j)} \) such that \( x_i = (u_i, v_i) \). Let \( e \in R \) be the idempotent such that \( \text{Ann}(v_i) = R(1 - e) \) and put \( y_{n+1} = ex_i \). Then \( y_i, i \leq n + 1 \) are independent and \( \sum_{i < \chi_0} R y_i \) is a direct summand of \( F_C \). Hence, $\text{WSPLIT}(I, F, \chi_0)$. To prove the second assertion, consider a set \( \{ e_i \mid i < \chi_0 \} \) of orthogonal idempotents of the ring \( R \). For \( n < \chi_0 \), let \( 1_n \) be the element of \( R^{\chi_0} \) such that \( 1_n \xi_i = 0 \) if \( i \neq n \), and \( 1_n \xi_n = 1 \). For \( i < \chi_0 \), put \( x_i = \sum_{n \leq i} e_{i-n} 1_n \). Then \( I_1 = R^{\chi_0} \) and \( I_2 = \sum_{i < \chi_0} R x_i \) provide the required examples.

3.12. Remark. Theorem 3.8 is formulated for the class \( \mathcal{C} \) of all regular rings such that each left ideal is countably generated. Clearly, \( \mathcal{C} \) contains all countable regular rings. Nevertheless, \( \mathcal{C} \) contains also all direct limits of countable directed systems of arbitrary simple completely reducible rings. Hence, \( \mathcal{C} \) contains rings of arbitrary infinite cardinality.

References


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