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ORTHOGONALITY SPACES AND ATOMISTIC ORTHOCOMPLEMENTED LATTICES

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0. INTRODUCTION

The purpose of this paper is to investigate orthogonality spaces and atomistic orthocomplemented lattices related to them. Especially we investigate conditions under which such an atomistic orthocomplemented lattice may be represented by the lattice of closed subspaces of a generalized inner product space. Some previously known conditions are compared with so called condition of minimal dependence or postulate of minimal superposition introduced by S. P. Gudder in [10] and used by S. Pulmannová in [22] and [23] to obtain representations of quantum logics and transition probability spaces. We show that it is possible to reduce the condition of minimal dependence to a condition of 3-minimal dependence. Similar weakened forms of the minimal superposition postulate have been studied by W. Guz [14] in orthomodular structures. We show the possibility of reduction in an atomistic orthocomplemented lattice corresponding to an orthogonality space. If the latter lattice is orthomodular, even the 3-minimal dependence condition can be reduced to a condition of 2-minimal dependence, which is equivalent to an atomic exchange property.

An important example of an orthogonality space is a generalized inner product space. The atomistic orthocomplemented lattice associated with this orthogonality space is just the lattice of closed subspaces. The orthomodularity of this lattice captivates a wide interest in the literature as it is equivalent to so called hilbertian property. In conclusion of our work, the orthomodularity of the atomistic orthocomplemented lattice associated with a special orthogonality space is characterized by means of so called splitting sets (cf. [24], [21], [11], [9], [3], and [4]).

1. DEFINITIONS AND PRELIMINARY RESULTS

In this preparatory section we define an orthogonality space $(S, \perp)$, closed sets and linear sets in $(S, \perp)$, and a reduction $(\tilde{S}, \perp)$ of $(S, \perp)$. We prove some basic facts about them and carry out some necessary observation on the lattice $L(S, \perp)$ of closed sets and on the lattice $K(S, \perp)$ of linear sets. Our general theory is supplemented by several concrete examples.

Still a remark concerning notation used in this article: we identify a one-element set $\{x\}$ with the element x.

Let S be a nonempty set. Recall that a *closure operation* on S is a map $A \mapsto \bar{A}$ on the subsets of S satisfying the following postulates:

(i) $A \subseteq \bar{A}$ for every subset A of S (extensiveness),

(ii) $\bar{A} = \bar{\bar{A}}$ for every subset A of S (idempotency),

(iii) $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$ for all subsets A, B of S (isotony).

A subset A of S is *closed* if $A = \bar{A}$. The collection of all closed subsets of S forms a complete lattice under set-inclusion in which

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i \quad \text{and} \quad \bigvee_{i \in I} A_i = \Big(\bigcup_{i \in I} A_i\Big)^-$$

where $\cap$ and $\cup$ denote the set-theoretical intersection and union, respectively. In addition, $\bar{\emptyset}$ is the smallest element and S is the greatest element in the lattice (see [2]).

Let S be a nonempty set. Let $\perp$ be a *symmetric* binary relation on S. For a subset A of S put

$$A' = \{y \in S \mid x \perp y \text{ for all } x \in A\} \quad \text{and} \quad \bar{A} = (A')'$$

Then the map $A \mapsto \bar{A}$ is a closure operation on S and the set

$$L(S, \perp) = \{A \subseteq S \mid A = \bar{A}\}$$

of all closed subsets of S is a complete lattice with the smallest element S' and the greatest element S. Moreover, the map $A \mapsto A'$ is an involutive dual automorphism of $L(S, \perp)$ (see [2]).

In what follows in this section we shall assume that S is a nonempty set equipped with a symmetric binary relation $\perp$. When is the unary operation $A \mapsto A'$ an orthocomplementation on $L(S, \perp)$ is solved in our first lemma (see also [2]).

1.1. Lemma. *The lattice $L(S, \perp)$ is orthocomplemented if and only if $\perp$ has the following property:*

(1) *For every $x \in S$, $x \perp x$ implies $x \perp y$ for all $y \in S$.*

Proof. The lattice $L(S, \perp)$ is orthocomplemented if $A \wedge A' = S'$ (or dually, if $A \vee A' = S$) for all $A \in L(S, \perp)$. To prove necessity, assume that $L(S, \perp)$ is orthocomplemented, $x \in S$ and $x \perp x$. Since $x' \in L(S, \perp)$, we get $x \in x' \wedge \bar{x} = S'$ and thus $x \perp y$ for all $y \in S$. Conversely, if (1) holds and if $A \in L(S, \perp)$ then every element $x \in A \wedge A'$ satisfies $x \perp x$ which means that $x \in S'$. Since from $A, A' \subseteq S$ we obtain $S' \subseteq A \wedge A'$, this gives $A \wedge A' = S'$. □

It is easy to see that every atom in $L(S, \perp)$ is of the form $\bar{x}$ for some $x \in S - S'$. In general, not every $\bar{x}$ $(x \in S - S')$ is an atom in $L(S, \perp)$. Since

$$A = \bigvee(\bar{x} \mid x \in A - S')$$

for every $A \in L(S, \perp)$, the presumption that every $\bar{x}$ $(x \in S - S')$ is an atom in $L(S, \perp)$ guarantees that the lattice $L(S, \perp)$ will be atomistic. Our second lemma brings several useful conditions each of them is equivalent to the statement: every $\bar{x}$ $(x \in S - S')$ is an atom in $L(S, \perp)$.

1.2. Lemma. *The following four conditions are equivalent:*

(2) *If $x, y \in S$ and $y \notin S'$, then $x' \subseteq y'$ implies $x' = y'$.*

(2a) *For every $x \in S$, $\bar{x} = \{y \in S \mid x' = y'\} \cup S'$.*

(2b) *The set $\{\bar{x} \mid x \in S - S'\}$ is the set of atoms of $L(S, \perp)$.*

(2c) *The set $\{x' \mid x \in S - S'\}$ is the set of coatoms of $L(S, \perp)$.*

Proof. (2) $\Rightarrow$ (2a) If $x \in S$ then every element $y \in \bar{x} - S'$ satisfies $x' \subseteq y'$ which implies $x' = y'$ by condition (2). Thus $\bar{x} \subseteq \{y \in S \mid x' = y'\} \cup S'$ and because the opposite inclusion is evident, we obtain the desired equality.

(2a) $\Rightarrow$ (2b) Let $x \in S - S'$ and let $A \in L(S, \perp)$ be such that $S' \subset A \subseteq \bar{x}$. Choose an element $y \in A - S'$. According to condition (2a) it holds $x' = y'$ which means that $\bar{x} = \bar{y} = A$. Therefore $\bar{x}$ is an atom in $L(S, \perp)$. This is enough for validity of (2b).

(2b) $\Rightarrow$ (2c) The implication follows by duality.

(2c) $\Rightarrow$ (2) Assume that $x, y \in S$ are such that $y \notin S'$ and $x' \subseteq y'$. Then x does not belong to S' and hence by condition (2c), x' is a coatom in $L(S, \perp)$. Therefore $x' = y'$. □

1.3. Definition. Let S be a nonempty set endowed with a symmetric binary relation $\perp$. We shall call a couple $(S, \perp)$ satisfying conditions (1) and (2), an *orthogonality space* (compare with [6], [16], [18]).

1.4. Proposition. *Let $(S, \perp)$ be an orthogonality space. Then $L(S, \perp)$ is a complete, atomistic orthocomplemented lattice.*

Proof. The lattice $L(S, \perp)$ is orthocomplemented by Lemma 1.1 and atomistic by Lemma 1.2. □

Let S be a nonempty set and let $\perp$ be a symmetric binary relation on S. A subset A of S will be called *linear* if $\{x, y\}^- \subseteq A$ for every $x, y \in A$. Denote by $K(S, \perp)$ the set of linear subsets of S. For $A \subseteq S$ we denote by the symbol kA the smallest linear subset of S containing A. Then the set of linear subsets of S can be expressed as

$$K(S, \perp) = \{A \subseteq S \mid A = kA\}\,.$$

It is evident that $L(S, \perp) \subseteq K(S, \perp)$. For $A, B \in K(S, \perp)$ we shall use the following notation: $A + B = k(A \cup B)$.

1.5. Proposition. *The map $A \mapsto kA$ is a closure operation on S. Consequently, the set $K(S, \perp)$ is a complete lattice with the meet $\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i$ and the join $\sum_{i \in I} A_i = k(\bigcup_{i \in I} A_i)$. The empty set $\emptyset$ is the smallest element and S is the greatest element of $K(S, \perp)$.*

We continue perceiving the linear subsets of S. Evidently, $k\{x, y\} = \{x, y\}^-$ and

$kA \subseteq \bar{A}$, where $x, y \in S$ and $A \subseteq S$. Let us observe that for every nonempty set $A \in K(S, \perp)$ we have $S' \subseteq A$. In what follows we shall consider S' as the smallest element of $K(S, \perp)$. With this supposition, the set of atoms (and dually, the set of coatoms) in $K(S, \perp)$ is equal to the set of atoms (the set of coatoms) in $L(S, \perp)$. Note that if condition (2) is satisfied then $K(S, \perp)$ is an atomistic complete lattice with the atoms $kx = \bar{x}$ $(x \in S - S')$.

Recall that an atomistic lattice L is *compactly atomistic* if for every atom $x \in L$ and any set of atoms $A \subseteq L$ such that $x \leqq \bigvee(y \mid y \in A)$ there is a finite set $F \subseteq A$ such that $x \leqq \bigvee(y \mid y \in F)$.

1.6. Proposition. *For every* $A \subseteq S$,

$$kA = \bigcup(kF \mid F \subseteq A, F \text{ is finite}).$$

Proof. Put $B = \bigcup(kF \mid F \subseteq A, F \text{ is finite})$. Clearly, $A \subseteq B \subseteq kA$. We see that it is enough to prove that $B = kB$. If $x, y \in B$ then there exist finite sets $F, G \subseteq A$ such that $x \in kF$ and $y \in kG$. Hence $\{x, y\}^- \subseteq k(F \cup G) \subseteq B$ and thus $B = kB$. □

Corollary. *If* $(S, \perp)$ *has the property* (2) *then the lattice* $K(S, \perp)$ *is compactly atomistic.*

Before bringing into some examples, we introduce a procedure which, in a natural fashion, produces from a given orthogonality space $(S, \perp)$ a new orthogonality space $(\tilde{S}, \perp)$, called *the reduction of* $(S, \perp)$, having more powerful properties and possessing the same structure of closed and linear sets.

1.7. Lemma. *The following two conditions are equivalent:*

(3) *If* $x, y \in S$ *and* $y \notin S'$, *then* $x' \subseteq y'$ *implies* $x = y$.

(3a) *For every* $x \in S$, $\bar{x} = x \cup S'$.

Proof. (3) ⇒ (3a) Having an element $x \in S$, the inclusion $x \cup S' \subseteq \bar{x}$ is clear. Conversely, every element $y \in \bar{x} - S'$ satisfies $x' \subseteq y'$, hence $x = y$ by (3). Thus $\bar{x} \subseteq x \cup S'$.

(3a) ⇒ (3) Let $x, y \in S$ be such that $y \notin S'$ and $x' \subseteq y'$. Then $\bar{y} \subseteq \bar{x}$ which by (3a) implies $y \cup S' \subseteq x \cup S'$, and thus $y = x$. □

Looking at the conditions (2) and (3), we see that (3) implies (2). An interesting property, stronger than condition (1), and in the presence of which $S' = \emptyset$, is the following one:

(1°) $x \perp x$ for no $x \in S$ (anti-reflexivity).

Let us observe that provided $S' = \emptyset$, conditions (2) and (3) have the following forms, respectively:

(2°) If $x, y \in S$ then $x' \subseteq y'$ implies $x' = y'$.

(3°) If $x, y \in S$ then $x' \subseteq y'$ implies $x = y$.

Again, let S be a nonempty set with a symmetric binary relation $\perp$. Assume that $S' \neq S$. We shall use the following notation. For any subset A of S, put

$$\tilde{A} = \{\bar{x} \mid x \in A - S'\}.$$

Define $\tilde{x} \perp \tilde{y}$ when $x \perp y$ $(x, y \in S - S')$. It is easy to see that this rule defines a symmetric binary relation on $\tilde{S}$. For $\mathscr{A} \subseteq \tilde{S}$ put

$$\mathscr{A}^{\perp} = \{B \in \tilde{S} \mid A \perp B \text{ for all } A \in \mathscr{A}\} .$$

Our interest concentrates now upon the new system $(\tilde{S}, \perp)$ which reliably reflects the original structure of $(S, \perp)$ as the following results show.

1.8. Lemma. *For every subset A of S it holds:*
(i) $\tilde{A}^{\perp} = (A')^{\sim}$,
(ii) $(kA)^{\sim} = k\tilde{A}$.

Proof. (i) If $A \subseteq S$ then

$$\begin{aligned}\tilde{A}^{\perp} &= \{\tilde{y} \mid y \in S - S' \text{ and } x \perp y \text{ for all } x \in A - S'\} \\ &= \{\tilde{y} \mid y \in A' - S'\} = (A')^{\sim} .\end{aligned}$$

(ii) It is easy to see that for $A \subseteq S$,

$$\begin{aligned} kA &= \bigcup (A_n \mid n = 0, 1, 2, \ldots) \quad \text{where} \quad A_0 = A \quad \text{and} \\ A_n &= \bigcup (\{x, y\}^{-} \mid x, y \in A_{n-1}) \quad \text{for} \quad n \geqq 1 .\end{aligned}$$

Similarly,

$$\begin{aligned} k\tilde{A} &= \bigcup (\mathscr{A}_n \mid n = 0, 1, 2, \ldots) \quad \text{where} \quad \mathscr{A}_0 = \tilde{A} \quad \text{and} \\ \mathscr{A}_n &= \bigcup (\{B, C\}^{-} \mid B, C \in \mathscr{A}_{n-1}) \quad \text{for} \quad n \geqq 1 .\end{aligned}$$

Evidently, $\mathscr{A}_0 = \tilde{A}_0$. Suppose that $\mathscr{A}_n = \tilde{A}_n$ for some $n \geqq 0$. Then, according to (i),

$$\begin{aligned}\mathscr{A}_{n+1} &= \bigcup (\{B, C\}^{-} \mid B, C \in \mathscr{A}_n) \\ &= \bigcup (\{\tilde{x}, \tilde{y}\}^{-} \mid x, y \in A_n - S') \\ &= \bigcup ((\{x, y\}^{\sim})^{-} \mid x, y \in A_n - S') \\ &= \bigcup ((\{x, y\}^{-})^{\sim} \mid x, y \in A_n - S') \\ &= (\bigcup (\{x, y\}^{-} \mid x, y \in A_n - S'))^{\sim} = \tilde{A}_{n+1} .\end{aligned}$$

So, the induction gives $\mathscr{A}_n = \tilde{A}_n$ for all $n \geqq 0$. The rest of the proof is now clear. □

Using Lemma 1.8, it is easy to prove the following statement.

1.9. Proposition. *The maps $L(S, \perp) \to L(\tilde{S}, \perp)$, $A \mapsto \tilde{A}$, and $K(S, \perp) \to K(\tilde{S}, \perp)$, $A \mapsto \tilde{A}$, are lattice isomorphisms which carry the unary operation $A \mapsto A'$ onto the unary operation $\mathscr{A} \mapsto \mathscr{A}^{\perp}$.*

1.10. Lemma. *For all $A, B \in \tilde{S}$ it holds:*

$$A^{\perp} \subseteq B^{\perp} \quad \textit{implies} \quad B \subseteq A .$$

Proof. If $A, B \in \tilde{S}$ then $A = \tilde{x}$ and $B = \tilde{y}$ for some $x, y \in S - S'$. Then $A^{\perp} \subseteq B^{\perp}$ means that $\tilde{x}^{\perp} \subseteq \tilde{y}^{\perp}$ which is the same as $\tilde{x}^{\perp} \subseteq \tilde{y}^{\perp}$. By Lemma 1.8, this implies $(x')^{\sim} \subseteq$ $\subseteq (y')^{\sim}$ from which, using Proposition 1.9, it follows $x' \subseteq y'$. Hence $\bar{y} \subseteq \bar{x}$ and thus $B \subseteq A$. □

Corollary. *For all* $A, B \in \tilde{S}$ *the following implication is true*:

$$A^{\perp} = B^{\perp} \quad \textit{implies} \quad A = B\,.$$

1.11. Proposition. *Property* (1) *is equivalent to the anti-reflexivity in* $\tilde{S}$:

(1a) $A \perp A$ *for no* $A \in \tilde{S}$.

Property (2) *is equivalent to the following condition*:

(2d) *If* $A, B \in \tilde{S}$ *then* $A^{\perp} \subseteq B^{\perp}$ *implies* $A = B$.

Proof. (1) $\Rightarrow$ (1a) If $A \in \tilde{S}$ and $A \perp A$ then there is $x \in S - S'$ such that $A = \bar{x}$ and $x \perp x$, hence $x \in S'$ by (1), which is a contradiction.

(1a) $\Rightarrow$ (1) If $x \in S - S'$ and $x \perp x$ then $\bar{x} \in \tilde{S}$ and $\bar{x} \perp \bar{x}$ which is impossible by (1a).

(2) $\Rightarrow$ (2d) If $A, B \in \tilde{S}$ then $A = \bar{x}$ and $B = \bar{y}$ for some $x, y \in S - S'$. From $A^{\perp} \subseteq B^{\perp}$ we obtain $x' \subseteq y'$, as in the proof of Lemma 1.10. According to (2), $x' = y'$ and therefore $A = \bar{x} = \bar{y} = B$.

(2d) $\Rightarrow$ (2) Let $x, y \in S$ be such that $y \notin S'$ and $x' \subseteq y'$. Then $x \notin S'$ and $\bar{x}^{\perp} = \tilde{x}^{\perp} = (x')^{\sim} \subseteq (y')^{\sim} = \tilde{y}^{\perp} = \bar{y}^{\perp}$. By (2d), $\bar{x} = \bar{y}$ which means that $x' = y'$. □

Let $(S, \perp)$ be an orthogonality space. Using Proposition 1.11, we see that $(\tilde{S}, \perp)$ is also an orthogonality space, and satisfies (1°) and (3°). More precisely, we should say that $(\tilde{S}, \perp)$ satisfies conditions (1a) and (2d), which are $\tilde{S}$-analogous of properties (1°) and (3°), because (1°) and (3°) are expressed in terms of the set S, not $\tilde{S}$. We shall call the orthogonality space $(\tilde{S}, \perp)$, the *reduction* of the orthogonality space $(S, \perp)$. Let us note that adding S' to the elements of $\tilde{S}$ we obtain an orthogonality space satisfying (3).

1.12. Examples. (i) Let $(S, \leqq, 0, 1, {}^*)$ be a complete orthocomplemented lattice. For $x, y \in S$ define $x \perp y$ when $x \leqq y^*$. Then $\perp$ is a symmetric binary relation on S which satisfies (1). The lattice $L(S, \perp)$ consists of the principal ideals of S and thus is ortho-isomorphic with S.

(ii) Let $(L, \leqq, 0, 1, {}^*)$ be a complete, atomistic, orthocomplemented lattice. Let S be the set of atoms in L and for $x, y \in S$ define $x \perp y$ when $x \leqq y^*$. Then $\perp$ is a symmetric binary relation on S which satisfies (1°) and (3°). Moreover, the lattice $L(S, \perp)$ is ortho-isomorphic with L (see the preceding example and [18], Theorem 2.5).

In the following two examples we assume that V is a generalized inner product space. This means, V is a vector space over a division ring D with an involutive anti-automorphism $\alpha \mapsto \alpha^*$, and $\langle \cdot, \cdot \rangle$ is an Hermitian form (the generalized inner product).

(iii) Let $S = V$ and for $x, y \in V$ let us define $x \perp y$ when $\langle x, y \rangle = 0$. Then $\perp$ is a symmetric binary relation on V. If $x \in V$ then $x \perp x$ implies $x = 0$ and therefore (1) is satisfied. As known, $\bar{x} = Dx = \{\alpha x \mid \alpha \in D\}$ for every $x \in V$. Hence, if $x, y \in V$, $y \neq 0$ and $x' \subseteq y'$, then $\bar{y} \subseteq \bar{x}$ which implies $\bar{x} = \bar{y}$. Thus condition (2) is satisfied and $(S, \perp)$ is therefore an orthogonality space. If $1 + 1 \neq 0$ then (3) is not satisfied as for example $x' \subseteq (2x)'$ for every $x \in V$, but $x \neq 2x \neq 0$ if $x \neq 0$.

The reduction of $(S, \perp)$ is $(\tilde{S}, \perp)$ where $\tilde{S} = \{Dx \mid x \in V, x \neq 0\}$.

(iv) Let $S = \{x \in V \mid \langle x, x \rangle = 1\}$ be nonempty and let $x \perp y$ if and only if $\langle x, y \rangle = 0$ $(x, y \in S)$. Then $x \perp x$ for no $x \in S$, hence (1°) is satisfied. Since $\bar{x} = \{\alpha x \mid \alpha \in D, \alpha\alpha^* = 1\}$ for every $x \in S$, also (2°) is satisfied, but (3°) is in general not satisfied.

2. MINIMAL DEPENDENCE CONDITION AND REPRESENTABILITY

Our attention in this section is aimed at orthogonality spaces $(S, \perp)$ satisfying conditions (1°) and (3°). We shall present a variety of properties of $(S, \perp)$, each of them is equivalent to the condition of minimal dependence, which enable to represent the lattice $L(S, \perp)$ as the lattice of closed subspaces of a vector space with an Hermitian form.

At the beginning we recall some definitions.

A set P whose elements are called *points* is called a *projective space* if there exists a family of subsets of P called *lines* satisfying the following two conditions:

(P1) Every line contains at least two points, and two different points x, y are contained in just one line, which is denoted by $\{x, y\}^-$.

(P2) Let x, y, z be points which are not contained in one line. If u, v are different points such that $u \in \{x, y\}^-$ and $v \in \{y, z\}^-$ then there exists a point t in $\{x, z\}^- \cap \{u, v\}^-$.

(See [19], p. 67.)

A lattice with a smallest element 0 has the *covering property* if for every atom x and every element y, $x \wedge y = 0$ implies y is covered by $x \vee y$, in notation $y \prec x \vee y$. (Cf. [19], p. 31.) Dually, a lattice with a greatest element 1 has the *dual covering property* if for every coatom x and every element y, $x \vee y = 1$ implies $x \wedge y \prec y$.

A lattice L is *modular* if $(x \wedge y) \vee z = (x \vee z) \wedge y$ for all $x, y, z \in L$, $z \leqq y$. Modularity implies both covering properties.

Now let S be a nonempty set with a symmetric binary relation $\perp$. When $A \subseteq S$, we say that $x \in S$ *depends on* A if $x \in \bar{A}$. We say that x *depends minimally on* A if $x \in \bar{A}$ and $x \notin \bar{B}$ for every proper subset B of A.

By definition, $(S, \perp)$ satisfies the *condition of minimal dependence* if for every finite subset F of S, and for every element $x \in S$ depending minimally on F, it is true that

$$(x \cup G)^- \cap \bar{H} \neq \emptyset$$

for all nonempty subsets G, H of F such that $G \cap H = \emptyset$ and $G \cup H = F$ (see [10], where the minimal dependence condition is called the minimal superposition postulate).

We shall introduce a weakened form of the condition of minimal dependence. Let $n \geqq 2$ be a natural number. We say that $(S, \perp)$ satisfies the *condition of n-minimal dependence* if for every elements $x_1, \ldots, x_n$ of S and every element $x \in S$ depending minimally on $\{x_1, \ldots, x_n\}$, it holds

$$\{x, x_1\}^- \cap \{x_2, \ldots, x_n\}^- \neq \emptyset .$$

Evidently, if $(S, \perp)$ satisfies the n-minimal dependence condition, it satisfies also the m-minimal dependence condition for every m, $2 \leqq m \leqq n$.

Weakened forms of the condition of minimal dependence ("MSP reduced") in orthomodular posets have been studied in [14].

An element A of $L(S, \perp)$ is called *finite* if there is a finite subset F of S such that $A = \bar{F}$. Dually, an element A of $L(S, \perp)$ is called *cofinite* if A' is a finite element. The set of all finite elements in $L(S, \perp)$ will be denoted by the symbol $F(S, \perp)$, and $G(S, \perp)$ will denote the set of all finite and cofinite elements in $L(S, \perp)$.

2.1. Lemma. *The following three conditions are equivalent in an atomistic lattice*:

(i) *If x, y are distinct atoms then $x \vee y$ covers x and y (the atomic covering property).*

(ii) *If x, y, z are atoms and $x \neq z$ then $x \leqq y \vee z$ implies $y \leqq x \vee z$ (the atomic exchange property).*

(iii) *If x, y, z, t are atoms and $x \neq y$ then $x, y \leqq z \vee t$ implies $z, t \leqq x \vee y$.*

The proof of this lemma is a routine and therefore is omitted.

In the sequel, we shall need the following observation. Let S be a nonempty set with a symmetric binary relation $\perp$. Assume that $S' = \emptyset$ and (2°) is satisfied. Then $(S, \perp)$ satisfies the condition of 2-minimal dependence if and only if the lattice $L(S, \perp)$ has the atomic exchange property.

Let us also observe that if $(S, \perp)$ satisfies (3°) and $S' = \emptyset$, then in view of Lemma 1.7, $\bar{x} = x$ for every $x \in S$.

Now we are ready to prove the main theorem of this section.

2.2. Theorem. *Let $(S, \perp)$ be an orthogonality space satisfying (1°) and (3°). The following conditions are equivalent*:

(i) *The covering property holds in $L(S, \perp)$.*

(ii) *If A and B are in $G(S, \perp)$ and both cover $A \wedge B$ then $A \vee B$ covers both A and B.*

(iii) *$F(S, \perp)$ is a modular lattice.*

(iv) *$(S, \perp)$ satisfies the condition of minimal dependence.*

(v) *$(S, \perp)$ satisfies the condition of 3-minimal dependence.*

(vi) *The set S with lines $\{x, y\}^-$, $x, y \in S$, $x \neq y$, forms a projective space.*

(vii) *$K(S, \perp)$ is modular.*

(viii) (a) *For every* $A \in L(S, \perp)$ *and for every* $x \in S$, $A + x = A \vee x$, *i.e.* $A + x \in L(S, \perp)$.

(b) *For all* $x, y, z, t \in S$, $x + (\{x, y\}^- \wedge \{z, t\}^-) = \{x, y\}^- \wedge (x + \{z, t\}^-)$.

(ix) (b) *and* (c) *For every* $x_1, \ldots, x_n \in S$, $k\{x_1, \ldots, x_n\} = \{x_1, \ldots, x_n\}^-$.

(x) (b) *and* (d) *For every* $x, y, z \in S$, $k\{x, y, z\} = \{x, y, z\}^-$.

(xi) (e) *For every* $A, B \in K(S, \perp)$, $A + B = \bigcup (\{x, y\}^- \mid x \in A, y \in B)$.

(f) $(S, \perp)$ *satisfies the* 2-*minimal dependence condition.*

Proof. We shall prove the theorem in the following way. In the first place, we shall prove all successive implications from (i) up to (x), and then we shall show that (x) ⇒ (v) and that (vi) ⇒ (xi) ⇒ (vii) ⇒ (i).

(i) ⇒ (ii) The implication is clear from [19], Theorem (7.10).

(ii) ⇒ (iii) The implication follows by [18], Lemma 4.2.

(iii) ⇒ (iv) Let us assume that an element $x \in S$ depends minimally on a finite subset F of S and that G, H are nonempty disjoint subsets of F such that $G \cup H = F$. Then, using modularity, we obtain:

$$x \in (x \cup G)^- \cap (\bar{G} \vee \bar{H}) = \bar{G} \vee ((x \cup G)^- \cap \bar{H}) .$$

Therefore $(x \cup G)^- \cap \bar{H}$ must be nonempty.

(iv) ⇒ (v) The implication is evident.

(v) ⇒ (vi) The lattice $L(S, \perp)$ satisfies the condition (iii) of Lemma 2.1, hence (P1), the first property of a projective space. To prove (P2), assume that $x, y, z, u, v \in S$ are such that $u \neq v$, $u \in \{x, y\}^-$ and $v \in \{y, z\}^-$. We are to show that $\{x, z\}^- \cap \{u, v\}^-$ is nonempty. Suppose that $x \neq u$. Hence we get $y \in \{x, u\}^-$ and thus $v \in \{x, z, u\}^-$. We can assume that the element v depends minimally on the set $\{x, z, u\}$ since otherwise, $v \in \{x, u\}^-$ implies $x \in \{u, v\}^-$ and $v \in \{z, u\}^-$ implies $z \in \{u, v\}^-$. An application of 3-minimal dependence condition gives the desired result.

(vi) ⇒ (vii) The implication follows by [19], Theorem (16.3).

(vii) ⇒ (viii) The part (b) follows immediately by modularity of the lattice $K(S, \perp)$. So, we shall prove condition (a). As $K(S, \perp)$ is modular, the covering property and the dual covering property hold in $K(S, \perp)$. Therefore, if $A \in L(S, \perp)$ and $x \in S - A$, then $A \prec A + x$ and $A' \wedge x' \prec A'$. Since the lattice $K(S, \perp)$ is atomistic, $A' = (A' \wedge x') + y$ for some $y \in S$, from which we obtain by the dual covering property,

$$A = (A')' = ((A' \wedge x') + y)' = (A \vee x) \wedge y' \prec A \vee x .$$

As $A + x \subseteq A \vee x$ and A is covered by both $A + x$ and $A \vee x$, we get $A + x = A \vee x$. (The proof is similar to that used in [20], p. 55.)

(viii) ⇒ (ix) Condition (b) is immediate and (c) follows from (a) by an easy induction.

(ix) ⇒ (x) The implication is evident.

(x) ⇒ (v) Let $x, y, z, t \in S$ be such that the element t depends minimally on the set $\{x, y, z\}$. Then $x + \{y, z\}^- = k\{x, y, z\} = \{x, y, z\}^-$ and hence

$$\begin{aligned} x + (\{t, x\}^- \cap \{y, z\}^-) &= \{t, x\}^- \cap (x + \{y, z\}^-) \\ &= \{t, x\}^- \cap \{x, y, z\}^- \\ &= \{t, x\}^- \end{aligned}$$

from which it follows that

$$\{t, x\}^- \cap \{y, z\}^- \neq \emptyset .$$

(vi) $\Rightarrow$ (xi) By an application of [19], Lemma (16.2) we obtain the part (e). The part (f) is implied by Lemma 2.1.

(xi) $\Rightarrow$ (vii) Let A, B, C be elements of $K(S, \perp)$ and let $B \subseteq C$. We have to prove that $(A + B) \cap C \subseteq (A \cap C) + B$. If $z \in (A + B) \cap C$ then $z \in \{x, y\}^-$ for some $x \in A$ and $y \in B$ by condition (e). We can assume that $y \neq z$, hence by condition (f) we get $x \in \{y, z\}^- \subseteq C$, and thus $\{x, y\}^- \subseteq (A \cap C) + B$ which concludes the proof. (The proof is similar to that used in [19], Theorem (16.3), (II).)

(vii) $\Rightarrow$ (i) Modularity of the lattice $K(S, \perp)$ implies that the covering property holds in $K(S, \perp)$. We have already proved that $A + x = A \vee x$ for every $A \in L(S, \perp)$ and every element $x \in S$. Therefore the covering property holds also in $L(S, \perp)$.

The proof is complete. □

Remark. If $(S, \perp)$ is an orthogonality space satisfying (1°) and (3°) then each of the conditions (i)–(xi) in Theorem 2.2 implies the following condition (MacLaren's condition):

For every $x, y \in S$, $x \neq y$ implies $x' \cap \{x, y\}^-$ is nonempty.

Proof. We prove that MacLaren's condition follows from (i). First observe that by duality, the lattice $L(S, \perp)$ has the dual covering property whenever it has the covering property. So, if we assume (i) and if $x, y \in S$, $x \neq y$, then by the dual covering property, $x' \cap \{x, y\}^- \prec \{x, y\}^-$. As $\{x, y\}^-$ is not an atom in $L(S, \perp)$, we have

$$x' \cap \{x, y\}^- \neq \emptyset .$$ □

To recognize representability of the lattice $L(S, \perp)$ of closed sets in an orthogonality space $(S, \perp)$, we need still express irreducibility of $L(S, \perp)$ as a property of $(S, \perp)$. Such a property is provided by the following lemma.

First recall that a lattice with 0 and 1 is irreducible when 0 and 1 are the only central elements in L. Further, recall that an element x of an orthocomplemented lattice $(L, \wedge, \vee, 0, 1, {}^*)$ is central if and only if $y = (x \wedge y) \wedge (x^* \wedge y)$ for all $y \in L$ (cf. [19], Lemma (29.9)).

2.3. Lemma. *Let $(S, \perp)$ be an orthogonality space. The lattice $L(S, \perp)$ is irreducible if and only if whenever A, B are subsets of S properly containing S' such that $A \cup B = S$, there exist elements $x \in A$ and $y \in B$ for which $x \perp y$ is not true.*

Proof. Assume that A, B are subsets of S properly containing S' such that $A \cup B = S$ and $x \perp y$ for all $x \in A$ and $y \in B$. Using condition (1) we obtain $A \cap B = S'$ and then $A = B'$ and $B = A'$. Whence $A, B \in L(S, \perp)$. Let E be an arbitrary element of $L(S, \perp)$. Denote $C = A \cap E$ and $D = A' \cap E$. Then

$$E = C \cup D = (A \wedge E) \vee (A' \wedge E).$$

This means that A is a central element in the lattice $L(S, \perp)$ and therefore $L(S, \perp)$ is not irreducible.

Conversely, assume that $L(S, \perp)$ contains a central element A distinct from S' and S. To violate the above condition it is enough to prove that $A \cup A' = S$. Choose an element $x \in S - A$. If $y \in \bar{x} \cap A$ and $y \notin S'$ then $x' \subseteq y'$ which implies $x' = y'$ by condition (2). So, we have $x \in \bar{x} = \bar{y} \subseteq A$, a contradiction. Therefore $\bar{x} \cap A = S'$. Since A is central,

$$\bar{x} = (\bar{x} \wedge A) \vee (\bar{x} \wedge A') = \bar{x} \wedge A'.$$

Thus $x \in A'$. This gives $A \cup A' = S$. □

The following representability theorem holds (see [19], Theorem (34.5)).

Theorem. *Let L be an irreducible complete atomistic orthocomplemented lattice with the covering property and of length ≧4. There exists a division ring D with an involutive anti-automorphism and there exists a vector space V over D with an Hermitian from such that L is ortho-isomorphic to the lattice of closed subspaces of V.*

By [19], Theorem (34.2), a converse is also true: the lattice of closed subspaces of V, V as in the preceding theorem, is an irreducible complete atomistic orthocomplemented lattice with the covering property.

Let $(S, \perp)$ be an orthogonality space satisfying (1°) and (3°). Assume that $L(S, \perp)$, the lattice of closed sets in $(S, \perp)$, is irreducible, i.e. that $(S, \perp)$ satisfies the condition of Lemma 2.3, and that $L(S, \perp)$ is of length ≧4. Theorem 2.2 yields a series of equivalent sufficient conditions under which $L(S, \perp)$ can be represented by the lattice of closed subspaces of V, V as in the preceding theorem. These conditions are also necessary.

3. ORTHOMODULARITY

Our interest in this section is oriented to a study of orthogonality spaces $(S, \perp)$ satisfying (1°), (3°) and the conditions (i)–(xi) of Theorem 2.2, which is related to orthomodularity occurring inside the lattice $L(S, \perp)$.

First, we shall observe that Theorem 2.2 has a continuation concerning orthomodularity (see Theorem 3.2 below). We shall define splitting sets in $(S, \perp)$ and present several statements about them, especially that the poset of all splitting sets is orthomodular and that $L(S, \perp)$ is its cut-completion. Finally, we shall formulate some conditions based on splitting sets under which $L(S, \perp)$ is orthomodular.

Recall that an *orthomodular lattice* is an orthocomplemented lattice satisfying the orthomodular law,

(i) $x \leqq y$ implies $x \vee (x^* \wedge y) = y$,

or equivalently, satisfying the following condition,

(ii) $x \leqq y$ and $x^* \wedge y = 0$ implies $x = y$.

An *orthomodular poset* is an orthocomplemented poset in which $x \vee y$ exists whenever $x \leqq y^*$ and in which (i) or (ii) is satisfied. Every interval $[0, x]$ in an orthomodular lattice is also an orthomodular lattice with the *relative orthocomplementation* $y \mapsto y^* \wedge x$ (see [17]).

Let S be a nonempty set with a symmetric binary relation $\perp$. *A* subset A of S is called *orthogonal* if $x \perp y$ for all $x, y \in A$, $x \neq y$. We say that elements $x_1, \ldots, x_n \in S$ are *independent* if

$$x_i \notin \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}^-$$

for every $i \in \{1, \ldots, n\}$.

3.1. Lemma. *If x, y, z are atoms in an atomistic orthomodular lattice having the atomic covering property, then the elements $(x \vee y) \wedge y^*$ and $(x \vee y \vee z) \wedge (y \vee z)^*$ are atoms or* 0.

Proof. In the first part of the proof assume that x, y are distinct atoms. By orthomodularity, $s = (x \vee y) \wedge y^* \neq 0$. Let z be an atom under s. Then $y \neq z$ and $s < x \vee y$ from which, by the atomic covering property, it follows $z \prec y \vee z = x \vee y$, hence $s = z$. This means that the element s is an atom.

Secondly, let x, y, z be atoms. Using the fact that $\{y^*, x \vee y, y \vee z\}$ is a Greechie set (see [17]), we obtain

$$\begin{aligned} w &= (x \vee y \vee z) \wedge (y \vee z)^* \\ &= z^* \wedge y^* \wedge ((x \vee y) \vee (y \vee z)) \\ &= z^* \wedge ((y^* \wedge (x \vee y)) \vee (y^* \wedge (y \vee z))) \\ &= z^* \wedge (u \vee v)\,, \end{aligned}$$

where $u = y^* \wedge (x \vee y)$ and $v = y^* \wedge (y \vee z)$. We can suppose that $x \neq y \neq z$, hence u and v are atoms by the preceding part of the proof. If $w = u \vee v$ then $v \leqq z^*$ which implies $v = 0$, a contradiction. Assume therefore that $0 \neq w < u \vee v$ and $u \neq v$. Let t be an atom under w. As $t \neq v$, we get $t \prec t \vee v = u \vee v$ and thus $w = t$, w is an atom. □

Now we can prove the promised continuation of Theorem 2.2.

3.2. Theorem. *Let $(S, \perp)$ be an orthogonality space satisfying* (1°) *and* (3°). *The conditions* (i)–(xi) *of Theorem* 2.2 *are equivalent to the following condition*:

(xii) (*f*) $(S, \perp)$ *satisfies the* 2-*minimal dependence condition.*

(g) *For every independent elements $x, y, z \in S$, the lattice $[\emptyset, \{x, y, z\}^-]$ endowed with the unary operation $A \mapsto A' \cap \{x, y, z\}^-$ is orthomodular.*

Proof. First assume that the conditions of Theorem 2.2 are satisfied. We prove that for every $F \in F(S, \perp)$, the lattice $L = [\emptyset, F]$ endowed with the unary operation $A \mapsto A' \cap F$ is orthomodular. Since $F(S, \perp)$ is an ideal in the lattice $L(S, \perp)$ (see [18], Theorem 4.1), $L \subseteq F(S, \perp)$. Let us observe that $A + B = A \vee B$ for every $A \in L(S, \perp)$ and $B \in F(S, \perp)$. Using modularity of the lattice $K(S, \perp)$, we obtain for every $A \in L$,

$$(A' \cap F)' \cap F = (A \vee F') \cap F = (A + F') \cap F = A + (F' \cap F) = A\,,$$

which shows that L is orthocomplemented. Since L is modular, it is also orthomodular.

Conversely, assume that (xii) is satisfied. We prove that $(S, \perp)$ satisfies the condition of 3-minimal dependence. Suppose that an element $t \in S$ depends minimally on a set $\{x, y, z\} \subseteq S$. It is to show that $G \cap H$ is nonempty, where $G = \{x, t\}^-$ and $H = \{y, z\}^-$. With respect to (f), which is the 2-minimal dependence condition, we can assume that the elements x, y, z are independent. Let us denote $A = \{x, y, z\}^-$, $B = \{x, y, t\}^-$, $C = \{x, y\}^-$, $D = A \cap C'$, $E = B \cap G'$ and $F = A \cap H'$. It is clear that $B \subseteq A$. We consider the orthomodular lattice $[\emptyset, A]$. By Lemmas 2.1 and 3.1, D, E and F are atoms or $\emptyset$. If $D \subseteq t'$ then $t \in A \cap D' = C$ which contradicts the minimal dependence of t. Threfore

$$A \cap B' = A \cap C' \cap t' = D \cap t' = \emptyset\,,$$

from which it follows $B = A$. Suppose to the contrary that $G \cap H = \emptyset$. Then

$$A = A \cap (G \cap H)' = (A \cap G') \vee (A \cap H') = E \vee F\,.$$

By condition (iii) of Lemma 2.1, $C = E \vee F = A$ which is impossible since x, y, z are independent. We conclude that $G \cap H$ must be nonempty. □

Corollary. *In a complete, atomistic, orthomodular lattice, the atomic covering property is equivalent to the covering property.*

Proof. It suffices to use Examples 1.12, (ii). □

It is interesting to study conditions, this means properties of an orthogonality space $(S, \perp)$, under which the lattice $L(S, \perp)$ is orthomodular provided the conditions of Theorems 2.2 and 3.2 are satisfied.

A general characterization of orthomodularity of $L(S, \perp)$ is given in [6] (see also [7]). It can be formulated as follows. Let S be a nonempty set with a symmetric binary relation $\perp$. Assume that $(S, \perp)$ satisfies condition (1), i.e. that the lattice $L(S, \perp)$ is orthocomplemented. Then $L(S, \perp)$ is orthomodular if and only if for every $A \in L(S, \perp)$, $A = \bar{B}$ for each maximal orthogonal subset B of A.

An important example of an orthogonality space is an inner product space (real or complex), where $\perp$ is defined by $x \perp y$ if and only if $\langle x, y\rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the inner product (see Examples 1.12, (iii) and (iv)).

The orthomodularity of the lattice $L(S, \perp)$ of an inner product-orthogonality space $(S, \perp)$ has been often studied in the literature. In [1], it is shown that $L(S, \perp)$

is orthomodular if and only if the inner product space is complete, i.e. a Hilbert space (see also [19], Theorem (34.9)). The problem is by no means easy, as R. Goldblatt [8] has shown, orthomodularity of $L(S, \perp)$ can not be expressed by any first-order properties of $(S, \perp)$. A series of other conditions (based prevailingly on states and splitting subspaces) characterizing the orthomodularity of the lattice $L(S, \perp)$ of an inner product space is presented in works [11], [9], [15], [3], [4], [5]. We shall formulate orthomodularity conditions using splitting subspaces in a more general context.

Let $(S, \perp)$ be an orthogonality space satisfying (1°) and (3°). A subset A of S is called *splitting* if $A \in K(S, \perp)$ and $A + A' = S$. Let us denote by $E(S, \perp)$ the set of splitting subsets of S. Using Lemmas 1.2 and 1.7 we obtain that for every $x \in S$, x is an atom in $K(S, \perp)$ and x' is a coatom in $K(S, \perp)$, hence $x \in E(S, \perp)$. We also see that $A' \in E(S, \perp)$ for every $A \in E(S, \perp)$. It is clear that $\emptyset$ and S are splitting sets.

More about $E(S, \perp)$ can be said when the lattice $K(S, \perp)$ is modular which is demonstrated by the next two results. As to the following proposition, see also [21] where splitting subspaces of a vector space with a linear orthogonality relation are dealt with.

3.3. Proposition. *Let $(S, \perp)$ be an orthogonality space satisfying (1°) and (3°). If $K(S, \perp)$ is modular then the following conditions are satisfied:*

(i) *If $A \in E(S, \perp)$, $B \in K(S, \perp)$, $A \subseteq B$ and $A' \cap B = \emptyset$, then $A = B$.*

(ii) *$E(S, \perp) \subseteq L(S, \perp)$.*

(iii) *$A + B = A \vee B \in E(S, \perp)$ for all $A, B \in E(S, \perp)$, $B \subseteq A'$.*

(iv) *$E(S, \perp)$ is an orthomodular poset.*

(v) *$A + x \in E(S, \perp)$ for every $A \in E(S, \perp)$ and $x \in S$.*

Proof. (i) By modularity we get $B = B \cap (A + A') = A + (B \cap A') = A$.

(ii) If $A \in E(S, \perp)$, it suffices to put $B = \bar{A}$ and to use (i).

(iii) Let $A, B \in E(S, \perp)$ and let $B \subseteq A'$. Modularity implies $S = A + (A' \cap (B + B')) = A + B + (A' \cap B') = A + B + (A + B)'$. Therefore $A + B$ belongs to $E(S, \perp)$. By (ii), $A + B \in L(S, \perp)$, hence $A + B = A \vee B$.

(iv) Using condition (ii), it is easy to check that $A \mapsto A'$ is an orthocompementation on $E(S, \perp)$. The rest follows by (i) and (iii).

(v) Assume that $A \in E(S, \perp)$ and $x \in S - A$. By condition (e) of Theorem 2.2, $x \in y + z$ for some $y \in A$ and $z \in A'$, hence by condition (f) of Theorem 2.2 we obtain $z \in x + y$. Therefore $A + x = A + z \in E(S, \perp)$ by (iii). □

In general, $E(S, \perp)$ need not be a lattice. However, if $E(S, \perp)$ is a lattice then it is a sublattice of $L(S, \perp)$. Namely, if $A, B \in E(S, \perp)$ and C is the meet of A, B in $E(S, \perp)$, then $C \subseteq A \cap B$ and since $x \in E(S, \perp)$ for every $x \in S$, $A \cap B \subseteq C$, and thus $C = A \cap B$. Using the orthocomplementation we get that the join of A, B in $E(S, \perp)$ is $A \vee B$.

The standard method of embedding a partially ordered set into a complete lattice uses completion by cuts. If the poset is orthocomplemented, the completion can be constructed by using the orthogonality relation (see [18]). Since S is a join dense subset in $E(S, \perp)$, a direct application of [18], Theorem 2.5 gives the following result.

3.4. Theorem. *Let* $(S, \perp)$ *be an orthogonality space satisfying* $(1°)$ *and* $(3°)$. *If* $K(S, \perp)$ *is modular then the cut-completion of* $E(S, \perp)$ *is ortho-isomorphic to* $L(S, \perp)$.

3.5. Theorem. *Let* $(S, \perp)$ *be an orthogonality space satisfying* $(1°)$ *and* $(3°)$ *and let the lattice* $K(S, \perp)$ *be modular. The following statements are equivalent:*

(i) *The lattice* $L(S, \perp)$ *is orthomodular.*

(ii) $E(S, \perp) = L(S, \perp)$.

(iii) $E(S, \perp)$ *is a complete lattice.*

(iv) $\bar{A} \in E(S, \perp)$ *for every orthogonal subset* A *of* S.

Proof. Using Theorem 2.2 and [19], Lemma (30.7), it can be easily derived that $L(S, \perp)$ is orthomodular if and only if $A + A' = S$ for every $A \in L(S, \perp)$. This means that (i) and (ii) are equivalent. It is clear that (ii) implies (iii) and (iv).

(iii) $\Rightarrow$ (ii) Let $A \in L(S, \perp)$. Since $E(S, \perp)$ is complete, there is $B \in E(S, \perp)$ smallest such that $A \subseteq B$. For every $x \in A'$ it holds $x' \in E(S, \perp)$ and $A \subseteq x'$, hence $B \subseteq x'$, which means that $x \in B'$. Thus $A' \subseteq B'$, i.e. $B \subseteq A$. Therefore $A = B \in E(S, \perp)$.

(iv) $\Rightarrow$ (ii) Let $A \in L(S, \perp)$ and let B be a maximal orthogonal subset of A. Evidently, $\bar{B} \subseteq A$. $\bar{B} \in E(S, \perp)$ means that $\bar{B} + B' = S$. Hence if $x \in A - \bar{B}$ then by condition (e) of Theorem 2.2, $x \in y + z$ for some $y \in \bar{B}$ and $z \in B'$. Using the atomic exchange property we get $z \in x + y \subseteq A$ which is impossible since B is maximal. Therefore $A = \bar{B} \in E(S, \perp)$. □

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