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EQUIVALENCE OF VON NEUMANN REGULAR AND IDEMPOTENT MATRICES

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1. INTRODUCTION

Let R be a ring (here: always with unity) and $A \in M_{m\times n}(R)$; if $X \in M_{n\times m}(R)$ is such that $AXA = A$ then X is said to be a *von Neumann regular inverse of* A (notation: $X \in A\{1\}$). In case A is von Neumann regular, an arbitrary von Neumann regular inverse of A will be denoted by $A^{(1)}$. The set of all X such that $AXA = A$ and $XAX = X$ will be denoted by $A\{1, 2\}$, and if such an X exists, it will be denoted by $\tilde{A}$. It can be shown that if A is von Neumann regular, it also has an $\tilde{A}$.

The diagonal matrix

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix} = \operatorname{diag}[d_1, d_2, \ldots, d_r]$$

will be denoted by dg_d.

A diagonal matrix which is von Neumann regular has a von Neumann regular inverse $\mathrm{dg}_d^{(1)} = \operatorname{diag}[d_1^{(1)}, d_2^{(1)}, \ldots, d_r^{(1)}]$. If in the diagonal matrix dg_d all diagonal elements are 1, then we denote this r by r unit matrix by 1_r. In [10] the following definition of an ID-ring was introduced:

Definition. A ring R is called an *ID-ring* iff every idempotent matrix over R is diagonalizable; i.e. for all $E \in M_n(R)$, $n \in \mathbb{N}$, $E^2 = E$ there exist invertible matrices $P, Q \in M_n(R)$ such that $PEQ = \mathrm{dg}_e = \operatorname{diag}[e_1, \ldots, e_r, 0, \ldots, 0]$, with $\mathrm{dg}_e = \mathrm{dg}_e^2 = \mathrm{dg}_e^T$.

It can be shown that an idempotent matrix E is equivalent to another idempotent matrix iff it is similar to that idempotent matrix; so, in the above definition we may suppose that $Q = P^{-1}$.

Any von Neumann regular matrix over an artinian ring or over an ID-domain, is equivalent to a diagonal idempotent matrix (see [4]). The class of ID-rings over which this remains true, will be extended. However, we first will consider the more general problem of the equivalence of a von Neumann regular matrix to an idempotent matrix (diagonal or not).

2. EQUIVALENCE OF VON NEUMANN REGULAR TO IDEMPOTENT MATRICES

The equivalence of von Neumann regular matrices to idempotent matrices is related to a cancellation law for modules. In accordance with existing definitions (see [2] and [11]) we call a ring R *semi-cancellative* iff it follows from $^nR = A \oplus B = C \oplus D$ and $A \cong C$ that $B \cong D$ for all right (finitely generated) R-modules A, B, C, and D over R and $n \in \mathbb{N}$.

Proposition. *Every square von Neumann regular matrix over a ring R is equivalent to an idempotent matrix iff R is semi-cancellative.*

Proof. $\Rightarrow$ Let $^nR = A \oplus B = C \oplus D$; take idempotent matrices E_A, E_B, E_C and E_D such that $A = E_A \,.\, {}^nR$, $B = E_B \,.\, {}^nR$, $C = E_C \,.\, {}^nR$ and $D = E_D \,.\, {}^nR$. $A \cong C$ means that matrices X' and Y' exist over R such that $E_A = X'Y'$, $Y'X' = E_C$. Let X be the matrix $X = E_AX'E_C$ and $Y = E_CY'E_A$. Then X will be an $n \times n$ von Neumann regular matrix; so there exist invertible matrices P and Q over R such that $X = PGQ$, $G^2 = G$ over r. If one takes $Q^{-1}GP^{-1} \in X\{1, 2\}$, then $E_A \,.\, {}^nR = \operatorname{Im} XY = \operatorname{Im} PGP^{-1}$. It follows that $E_A = TGT^{-1}$ for some invertible T. In the same way $E_A = S^{-1}GS$ for some S. So $E_B \cong 1 - G \cong E_D$ and B and D are isomorphic.

$\Leftarrow$ If $A \in M_n(R)$ has a von Neumann regular inverse $\tilde{A}$, then $A\tilde{A} \,.\, {}^nR \oplus (1 - A\tilde{A}) \,.\, {}^nR = {}^nR = \tilde{A}A \,.\, {}^nR \oplus (1 - \tilde{A}A) \,.\, {}^nR$. Since R is semi-cancellative, it follows from $A\tilde{A} \cong \tilde{A}A$ that $1 - A\tilde{A} \cong 1 - \tilde{A}A$; thus $1 - A\tilde{A} = XY$ and $1 - \tilde{A}A = YX$, for some X, Y over R. Then $A = (A + (1 - A\tilde{A})\, X(1 - \tilde{A}A))\, \tilde{A}A$ and $(A + (1 - A\tilde{A}) \,.\, X(1 - \tilde{A}A))^{-1} = \tilde{A} + (1 - \tilde{A}A)\, Y(1 - A\tilde{A})$. So A is equivalent to the idempotent matrix $\tilde{A}A$. ■

Corollary. *A square von Neumann regular matrix A is equivalent to a matrix $A\tilde{A}$, for some von Neumann regular inverse $\tilde{A}$, iff it is equivalent to all AX, for all $X \in A\{1\}$.*

Proof. If $A = PA\tilde{A}Q$ and $X \in A\{1\}$ then $\operatorname{Im} A = \operatorname{Im} A\tilde{A} = \operatorname{Im} AX$ so $A\tilde{A} = TAXT^{-1}$ for some invertible T. Thus $A = (PT)\, AX(T^{-1}Q)$. ■

In the proposition, only square matrices were considered .This is not an essential restriction. Indeed, if every square von Neumann regular matrix is equivalent to an idempotent matrix, then every rectangular matrix $A \in M_{m \times n}(R)$ will be equivalent to a matrix

$$F' = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}_{m,n}, \quad F^2 = F .$$

If, for example, $m < n$, then A can be completed by zero rows such that

$$\begin{bmatrix} A \\ 0 \end{bmatrix} \in M_n(R) . \text{ Then } [\tilde{A}\ 0] \begin{bmatrix} A \\ 0 \end{bmatrix} = PGP^{-1} ,$$

$$\begin{bmatrix} A \\ 0 \end{bmatrix} [\tilde{A}\ 0] = Q^{-1}GQ \quad \text{for an} \quad \tilde{A} \in A\{1, 2\}$$

and an idempotent $G = G^2$. From $[A^T 0]^T [\tilde{A} 0] = Q^{-1}GQ$ it follows that $Q^{-1}GQ = F'$ (in which $F = A\tilde{A}$) and thus $\tilde{A}A = PGP^{-1} = PQQ^{-1}GQQ^{-1}P^{-1} = (PQ)F'(PQ)^{-1}$. Hence: $A = A\tilde{A}\,.\,A\,.\,\tilde{A}A = (FAPQ[F^T\,0]^T + 1 - F)\,.\,[F\,0]\,.\,Q^{-1}P^{-1}$ and $(FAPQ[F^T\,0]^T + 1 - F)^{-1} = [F\,0]\,.\,Q^{-1}P^{-1}\tilde{A}F + 1 - F$.

Examples. 1. Consider a vector space $\mathscr{V}$ over a field F with a field F with a denumerable basis $(x_1, x_2, \ldots)$ over F (see [3] and [5]). With respect to this basis one can consider the (non finite) matrices

$$U = \begin{bmatrix} 0 & 1 & & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & & \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & 0 & \ldots \\ 1 & 0 & \ldots \\ 0 & 1 & \ldots \\ \vdots & \vdots & \end{bmatrix}.$$

U is von Neumann regular, and $V \in U\{1, 2\}$; now $UV = 1$, $VU = \operatorname{diag}[0, 1, 1, \ldots]$, and here $1 - UV = 0$ is not isomorphic to $1 - VU = \operatorname{diag}[1, 0, 0, \ldots]$. So the ring of linear transformations over the vector space $\mathscr{V}$ has von Neumann regular elements that are not equivalent to an idempotent matrix.

2. R. Puystjens and J. Van Geel gave the following example of a von Neumann regular matrix that is not equivalent to an idempotent matrix. Take $\mathscr{W} = F[x, y, \delta_0]$ where x, y are variables and δ_0 the derivation given by $xy - yx = 1$. $\mathscr{W}$ (the Weyl-algebra) is then a Noetherian simple domain.

Consider the matrix

$$A = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix};$$

it is von Neumann regular, but not equivalent an idempotent matrix. Using the proposition, we can deduce this result in another way.

Consider

$$\tilde{A} = \begin{bmatrix} -y & x \\ 0 & 0 \end{bmatrix};$$

then

$$A\tilde{A} = \begin{bmatrix} -xy & x^2 \\ -y^2 & yx \end{bmatrix}, \quad \tilde{A}A = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}, \quad 1 - A\tilde{A} = \begin{bmatrix} 1 + xy & -x^2 \\ y^2 & 1 - yx \end{bmatrix},$$

$$1 - \tilde{A}A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

G. S. Rinehart (see [9]) noticed that $(x^2, yx - 1)$ is not a principal ideal; hence $1 - A\tilde{A}$ is not free. However, the other three idempotent matrices are isomorphic to a unit matrix. The following decompositions are obtained:

$${}^2R := \begin{bmatrix} -xy & x^2 \\ -y^2 & yx \end{bmatrix} . {}^2R \oplus \begin{bmatrix} 1 + xy & -x^2 \\ y^2 & 1 - yx \end{bmatrix} . {}^2R = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} . {}^2R \oplus \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} . {}^2R.$$

The terms on the left of each direct sum are isomorphic, but not those on the right.

By the proposition, it follows that A cannot be equivalent to an idempotent matrix.

3. Any von Neumann regular matrix over a Dedekind domain is equivalent to an idempotent matrix, but not necessarily to a diagonal one.

Indeed, suppose R is a commutative Dedekind domain. It follows from the Steinitz-Chevalley theory, that if ${}^nR = A \oplus B = C \oplus D$ and $A \cong C$ then either

i) $A \cong C \cong {}^rR$, and thus $B \cong D \cong {}^{n-r}R$; or

ii) $A \cong C \cong {}^{r-1}R \oplus I$, and thus $B \cong D \cong {}^{n-r-2}R \oplus J$.

Hence, any von Neumann regular matrix A over a Dedekind domain is equivalent to

i) a diagonal idempotent matrix diag $[1_r, 0]$; or

ii) to a matrix of the form

$$\begin{bmatrix} 1_{r-1} & 0 & 0 \\ 0 & E & 0 \\ 0 & & 0 & 0 \end{bmatrix}, \quad \text{with } E \in M_{2\times 2}(R) \text{ idempotent, non-diagonalizable.}$$

3. DIAGONALIZATION OF VON NEUMANN AND IDEMPOTENT MATRICES

The following proposition considers the diagonalizability of a von Neumann regular matrix over an ID-ring. It is not supposed that the diagonal matrix to which the von Neumann regular matrix is equivalent, should be idempotent.

Proposition. *Let R be an ID-ring; then every von Neumann regular matrix $A \in M_{m\times n}(R)$ can be diagonalized iff for every pair of isomorphic diagonal idempotent matrices $\mathrm{dg}_e^2 = \mathrm{dg}_e \cong \mathrm{dg}_f = \mathrm{dg}_f^2$ a von Neumann regular matrix dg_x exists such that $\mathrm{dg}_e = K\,\mathrm{dg}_x\,\mathrm{dg}_x^{(1)}\,K^{-1}$, $\mathrm{dg}_f = L^{-1}\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x\,L$ for some $\mathrm{dg}_x^{(1)}$ of dg_x and invertible K, L.*

Proof. $\Rightarrow$ Suppose every von Neumann regular matrix can be diagonalized. If $\mathrm{dg}_e \cong \mathrm{dg}_f$ then there exist matrices X, Y over R such that $\mathrm{dg}_e = XY$, $YX = \mathrm{dg}_f$ or else: $\mathrm{dg}_e = (\mathrm{dg}_e X\,\mathrm{dg}_f)\,.\,(\mathrm{dg}_f Y \mathrm{dg}_e)$ and $\mathrm{dg}_f = (\mathrm{dg}_f Y \mathrm{dg}_e)\,.\,(\mathrm{dg}_e X\,\mathrm{dg}_f)$.

Now $\mathrm{dg}_e X\,\mathrm{dg}_f$ is von Neumann regular; so $\mathrm{dg}_e X\,\mathrm{dg}_f = P\,\mathrm{dg}_x\,Q$ wiht P, Q invertible. Thus dg_x will be von Neumann regular too. Then $Q^{-1}\,dg_x^{(1)}P^{-1} \in \mathrm{dg}_e X\,\mathrm{dg}_f\{1,2\}$. Hence

$$\begin{aligned} \operatorname{Im} X &= \operatorname{Im} \mathrm{dg}_e X\,\mathrm{dg}_f\,Q^{-1}\,\mathrm{dg}_x^{(1)}\,P^{-1} = \operatorname{Im} P\,\mathrm{dg}_x\,\mathrm{dg}_x^{(1)}\,P^{-1} = \\ &= \operatorname{Im} \mathrm{dg}_e X\,\mathrm{dg}_f\,\mathrm{dg}_f\,Y\mathrm{dg}_e = \operatorname{Im} \mathrm{dg}_e\,. \end{aligned}$$

So $\mathrm{dg}_e = K\,\mathrm{dg}_x\,\mathrm{dg}_x^{(1)}\,K^{-1}$. Similarly, $\mathrm{dg}_f = L^{-1}\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x\,L$.

$\Leftarrow$ Let $\tilde{A} \in A\{1,2\}$; there exist invertible matrices P and Q such that $A\tilde{A} = P\,\mathrm{dg}_e\,P^{-1}$, $\tilde{A}A = Q^{-1}\,\mathrm{dg}_f\,Q$, for R is an ID-ring. The given condition assures that $A\tilde{A} = PK\,\mathrm{dg}_x\,\mathrm{dg}_x^{(1)}\,K^{-1}P^{-1}$, $\tilde{A}A = Q^{-1}L^{-1}\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x\,LQ$. So

$$\begin{aligned} A &= A\tilde{A}A\tilde{A}A = PK\,\mathrm{dg}_x\,\mathrm{dg}_x^{(1)}\,K^{-1}P^{-1}AQ^{-1}L^{-1}\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x\,LQ = \\ &= (PK)\,\mathrm{dg}_x\,[\mathrm{dg}_x^{(1)}(PK)^{-1}\,A(LQ)^{-1}\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x + 1 - \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x]\,(LQ)\,. \end{aligned}$$

Now

$$[\mathrm{dg}_x^{(1)}(PK)^{-1}A(LQ)^{-1}\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x + 1 - \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x]\,.$$
$$.\,[\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x(LQ)\,\tilde{A}(PK)\,\mathrm{dg}_x + 1 - \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x] =$$
$$= \mathrm{dg}_x^{(1)}(PK)^{-1}A\,.\,\tilde{A}A\,.\,\tilde{A}(PK)\,\mathrm{dg}_x + 1 - \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x =$$
$$= \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x + 1 - \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x = 1$$

and

$$[\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x(LQ)\,\tilde{A}(PK)\,\mathrm{dg}_x + 1 - \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x]\,.$$
$$.\,[\mathrm{dg}_x^{(1)}(PK)^{-1}A(LQ)^{-1}\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x + 1 - \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x] =$$
$$= \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x(LQ)\,\tilde{A}\,.\,A\tilde{A}\,.\,A(LQ)^{-1}\,\mathrm{dg}_x^{(1)}\,\mathrm{dg}_x + 1 - \mathrm{dg}_x^{(1)}\,\mathrm{dg}_x = 1\,. \quad \blacksquare$$

Corollary 1. *Over a commutative ID-ring every von Neumann regular matrix can be diagonalized.*

Proof. Let R be a commutative ID-ring. A Steger (see [10]) has shown that any idempotent matrix is in that case equivalent to a diagonal idempotent matrix $\mathrm{diag}\,[e_1, \ldots, e_r, 0, \ldots, 0]$, on which the conditions that $e_1 \mid e_{1+1}$, $\forall i \in \{1, \ldots, r-1\}$, may be imposed.

If $A \in M_{m\times n}(R)$, and $\tilde{A} \in A\{1, 2\}$, invertible matrices P and Q exist such that $A\tilde{A} = P\,\mathrm{dg}_e\,P^{-1}$ with $\mathrm{dg}_e = \mathrm{diag}\,[e_1, \ldots, e_r, 0, \ldots, 0] = \mathrm{dg}^2$, $e_1 \mid e_{i+1}$, $\forall i \in \{1, \ldots, r-1\}$ and $\tilde{A}A = Q^{-1}\,\mathrm{dg}_f\,Q$ with $\mathrm{dg}_f = \mathrm{diag}\,[f_1, \ldots, f_s, 0, \ldots, 0] = \mathrm{dg}^2$, $f_1 \mid f_{i+1}$, $\forall i \in \{1, \ldots, s-1\}$. Suppose $r < s$; since $\mathrm{dg}_e = P^{-1}AQ^{-1}\,.\,Q\tilde{A}P$ and $\mathrm{dg}_f = Q\tilde{A}P\,.\,P^{-1}AQ^{-1}$, dg_e and dg_f are isomorphic idempotent matrices.

So

$$\begin{bmatrix} e_1 & & \\ & \ddots & \\ & & e_r \end{bmatrix} \text{ and } \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_s \end{bmatrix}$$

are isomorphic idempotent matrices too.

There exist $s \times s$ matrices X and Y such that

$$\begin{bmatrix} e_1 & & & \\ & \ddots & & \\ & & e_r & \\ & & & 0 \end{bmatrix} = XY, \quad YX = \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_s \end{bmatrix}. \text{ So } \det \begin{bmatrix} e_1 & & & \\ & \ddots & & \\ & & e_r & \\ & & & 0 \end{bmatrix} = 0$$

$$= \det X\,.\,\det Y = \det(\mathrm{diag}\,[f_1, \ldots, f_s]) = f_s, \text{ so } f_s = 0\,.$$

If $r > s$, then in the same way it is obtained that $e_r = 0$.

If $r = s$, there exist $s \times s$ matrices X' and Y' such that

$$\begin{bmatrix} e_1 & & \\ & \ddots & \\ & & e_r \end{bmatrix} = X'\,.\,Y', \quad Y'\,.\,X' = \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_s \end{bmatrix}$$

and thus: $e_r = \det(\mathrm{diag}\,[e_1, \ldots, e_r, 0, \ldots, 0]) = \det X'\,.\,\det Y' =$

$= \det(\mathrm{diag}[f_1, \ldots, f_s]) = f_s$. Since d iag $[e_1, \ldots, e_{r-1}, e_r]$ and $\mathrm{diag}[f_1, \ldots, f_{r-1}, f_r] = \mathrm{diag}[f_1, \ldots, f_{r-1}, e_r]$ are isomorphic, $\mathrm{diag}[e_1, \ldots, e_{r-1}, e_r, 1 - e_r]$ and $\mathrm{diag}[f_1, \ldots, f_r, 1 - e_r]$ are isomorphic too. But $\mathrm{diag}[e_r, 1 - e_r] \cong \mathrm{diag}[1, 0]$, so $\mathrm{diag}[e_1, \ldots, e_{r-1}, 1, 0] \cong \mathrm{diag}[f_1, \ldots, f_{r-1}, 1, 0]$. By repetition of the given argument: $f_r = e_r, f_{r-1} = e_{r-1}, \ldots, f_1 = e_1$.

So it follows from $\mathrm{dg}_e^2 = \mathrm{dg}_e \cong \mathrm{dg}_f = \mathrm{dg}_f^2$ that $\mathrm{dg}_e = \mathrm{dg}_f$ (so, $L = K = 1_r$ and $\mathrm{dg}_x = \mathrm{dg}_x^{(1)}$ in the proposition). Thus, every von Neumann regular matrix over R can be diagonalized. ■

The following property, which has been studied in several papers (see for example [1], [4] and the references given in these papers), is now a corollary of the above proposition.

Corollary 2. *If R is an ID-ring with* 0 *and* 1 *as its only idempotent elements, then every von Neumann regular matrix over R is diagonalizble.*

Proof. In this case all diagonal idempotent matrices are of the form 1_s. Hence if $1_s \cong 1_t$ then $1_s = XY$ and $1_t = YX$ for some $X \in M_{s \times t}(R)$ and $Y \in M_{t \times s}(R)$. So $1_s = X\, 1_t\, 1_t X^{-1}$ and of course $1_t = 1_t, 1_t$. Thus the conditions of the proposition are satisfied (Take $K = X$, $L = 1_t$ and $\mathrm{dg}_x = 1_t$). ■

From the proofs of the corollaries it also follows that in these two cases a von Neumann regular matrix is equivalent to diagonal idempotent matrix.

Applicatiins. 1. R. Puystjens and J. Van Geel have formulated the following conjecture: "If R is an ID-ring and A is a von Neumann regular matrix, then AX is equivalent to A, for each von Neumann regular inverse X of A."

If one considers the Weyl-algebra $\mathscr{W} = F[x, y, \delta_0]$ then $M_2(\mathscr{W})$ is a left and right principal ideal ring (see [6]). Every matrix over the ring $M_2(\mathscr{W})$ can thus be diagonalized (see [7]). In particular $M_2(\mathscr{W})$ is an ID-ring. The element

$$\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$$

of this ring is however not equivalent to an idempotent element although it is von Neumann regular. Thus, every von Neumann regular matrix over $M_2(\mathscr{W})$ is equivalent to diagonal matrix, but not necessarily an idempotent one.

This illustrates the given proposition and provides a counter example to the conjecture.

2. Let R be an arbitrary ring (with unity); an element $a \in R$ is called

- *von Neumann regular* iff there is an element $b \in R$ such that $aba = a$.
- *unit regular* iff there is a unit $u \in R$ (i.e. u has a two-sided inverse) such that $aua = a$.

A ring R is called *partially unit regular* (abbreviation p.u.r.) iff every regular element is unit regular. Hall, Hartwig, Katz and Newman have formulated the following open question (see [1]): "Does R being p.u.r. always imply that $M_{n \times n}(R)$ is p.u.r.?"

The answer to this question is negative: the Weyl-algebra $\mathscr{W}$ is a domain, so it is a p.u.r. ring. It was shown above that there exist a 2 by 2 von Neumann regular matrix A over $\mathscr{W}$ which is not equivalent to an idempotent matrix. Hence, $M_{2\times 2}(\mathscr{W})$ cannot be p.u.r. (if $AUA = A$ for some invertible U, then $UAUA = UA$ so UA is idempotent; thus $A = U^{-}(UA)$ and A would be equivalent to an idempotent matrix).

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