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SELF DUALITY OF THE SYSTEM OF INTERVALS OF
A PARTIALLY ORDERED SET

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1. INTRODUCTION

For a partially ordered set P we denote by $\text{Int } P$ the system of all intervals $[a, b] = \{x \in P: a \leq x \leq b\}$, where $a, b \in P$ and $a \leq b$, including the empty set. The system $\text{Int } P$ is partially ordered by the set-theoretical inclusion.

If P is a lattice, then $\text{Int } P$ is a lattice as well. In general, $\text{Int } P$ need not be a lattice.

In [1], the following theorem was presented:

(A) Let L be a finite lattice. Then $\text{Int } L$ is selfdual if and only if either (i) $\text{card } L \leq 2$, or (ii) $\text{card } L = 4$ and L has two atoms.

Next, in [1] the author proposed the problem whether there exists an infinite lattice L such that $\text{Int } L$ is selfdual.

In the present paper it will be shown that the answer to this problem is negative. Namely, the following result will be proved:

(B) Let P be a partially ordered set with $\text{card } P > 4$. Then the partially ordered system $\text{Int } P$ is not selfdual.

Some questions concerning $\text{Int } L$ (where L is a lattice) have been studied in the papers [2]–[9].

2. PROOF OF (B)

If Q is a partially ordered set and a, b, c are elements of Q , then by writing $a \vee b = c$ we express the fact that c is the least upper bound of the set $\{a, b\}$ in Q ; the meaning of $a \wedge b = c$ is the dual one. If a and b are incomparable, then we write $a \parallel b$; the fact that a is covered by b will be expressed by writing $a < b$.

Q is said to be *selfdual* if there exists a dual automorphism of Q . If f is a dual automorphism of Q and $a, b, c \in Q$, then

$$a \vee b = c \Leftrightarrow f(a) \wedge f(b) = f(c),$$

and dually.

In what follows, P denotes a partially ordered set. Let $X \in \text{Int } P$.

X is an atom of $\text{Int } P$ if and only if there is $a \in P$ with $X = \{a\}$.

Let $X = [a, b]$. Then X is a dual atom of $\text{Int } P$ if and only if there is $[u, v] \in \text{Int } P$ such that $[u, v] = P$, and either (i) $a = u$ and $b < v$, or (ii) $u < a$ and $b = v$.

Let $X = [a_1, a_2]$ and $Y = [b_1, b_2]$ belong to $\text{Int } P$. Then $X \wedge Y$ does exist in $\text{Int } P$ if and only if either $X \cap Y = \emptyset$ (and in this case $X \wedge Y = \emptyset$), or both $a_1 \vee b_1$ and $a_2 \wedge b_2$ exist in P (and then $X \wedge Y = [a_1 \vee b_1, a_2 \wedge b_2]$).

Similarly, $X \vee Y$ exists in $\text{Int } P$ if and only if both $a_1 \wedge b_1$ and $a_2 \vee b_2$ exist in P ; in such a case $X \vee Y = [a_1 \wedge b_1, a_2 \vee b_2]$.

In particular, if $[a, b] \in \text{Int } P$, then

$$[a, b] = \{a\} \vee \{b\}.$$

Next, if $a, b, c \in P$ and $a < b < c$, then

$$[a, b] \wedge [b, c] = \{b\}.$$

2.1. Lemma. *Assume that the system $\text{Int } P$ is selfdual. Then there are $u, v \in L$ with $a \leq v$ such that $P = [u, v]$.*

Proof. There exists a dual automorphism f of $\text{Int } P$. Since \emptyset is the least element of $\text{Int } P$, $f(\emptyset)$ must be the largest element of $\text{Int } P$. Clearly $f(\emptyset) \neq \emptyset$ and hence there is $[u, v] \in \text{Int } P$ such that $[u, v] = P$.

In proving (B) we proceed by way of contradiction; suppose that $\text{card } P > 4$ and that the partially ordered system $\text{Int } P$ is selfdual. Let f be a fixed dual automorphism of $\text{Int } P$. In view of 2.1 there are $u, v \in P$ such that $P = [u, v]$.

2.2. Lemma. *Let $[a, b] \in \text{Int } P$, $[a, b] \neq P$. Then there are dual atoms X_1 and X_2 in $\text{Int } P$ such that $X_1 \wedge X_2 = [a, b]$.*

Proof. There is $[a_1, b_1] \in \text{Int } P$ such that $f([a_1, b_1]) = [a, b]$. Put $f(\{a_1\}) = X_1$ and $f(\{b_1\}) = X_2$. Since $\{a_1\}$ and $\{b_1\}$ are atoms of $\text{Int } P$, both X_1 and X_2 are dual atoms of $\text{Int } P$. Next, $\{a_1\} \vee \{b_1\} = [a_1, b_1]$. By applying the dual automorphism f we obtain that $X_1 \wedge X_2 = [a, b]$.

2.3. Lemma. *Let $[a, b] \in \text{Int } P$, $a \neq b$, $a \neq u$, $b \neq v$. Then $u < a$ and $b < v$.*

Proof. Let X_1 and X_2 be as in 2.2. There are $c_i, d_i \in P$ ($i = 1, 2$) with $X_1 = [c_1, d_1]$ and $X_2 = [c_2, d_2]$. If $c_1 = c_2 = u$, then $a = u$, which is a contradiction. Thus, without loss of generality we can suppose that $c_1 = u$ and $c_2 \neq u$. Then $d_2 = v$, $d_1 < v$ and $u < c_2$. Since $X_1 \cap X_2 = [a, b]$, we infer that $a = c_2$ and $b = d_1$.

2.4. Lemma. *Let C be a chain in P . Then $\text{card } C \leq 4$.*

Proof. This is an immediate consequence of 2.3.

Denote $f(\{u\}) = X$ and $f(\{v\}) = Y$. Then

$$\{u\} \vee \{v\} = P, \quad \{u\} \wedge \{v\} = \emptyset,$$

whence

$$(1) \quad X \cap Y = \emptyset,$$

$$(2) \quad X \vee Y = P.$$

There are x_i and y_i in P ($i = 1, 2$) such that $X = [x_1, x_2]$ and $Y = [y_1, y_2]$. From the fact that X and Y are dual atoms of $\text{Int } P$ and from (1), (2) we infer that some of the following conditions is valid:

- (α) $x_1 = u, x_2 < v, u < y_1, y_2 = v, x_2 \parallel y_1$;
- (β) $y_1 = u, y_2 < v, u < x_1, x_2 = v, y_2 \parallel x_1$.

Next, let $z \in L, f(\{z\}) = [t_1, t_2]$. We have either

- (α_1) $t_1 = u$ and $t_2 < v$,

or

- (β_1) $u < t_1$ and $t_2 = v$.

From the relation $[u, z] \cap [z, v] = \{z\}$ we obtain

$$f([u, z]) \vee f([z, v]) = f(\{z\}).$$

Because of $[u, z] = \{u\} \vee \{z\}$ and the analogous relation for $[z, v]$, we get

- (3) $(f(\{u\}) \wedge f(\{z\})) \vee (f(\{z\}) \wedge f(\{v\})) = f(\{z\})$.

2.5. Lemma. *Assume that (α) and (α_1) are valid. Let $u \neq z$. Then $y_1 \leq t_2$.*

Proof. In view of (3) we have

- (4) $([u, x_2] \wedge [u, t_2]) \vee ([u, t_2] \wedge [y_1, v]) = [u, t_2]$.

Then

$$[u, x_2] \wedge [u, t_2] = [u, x_2 \wedge t_2].$$

Next,

$$[u, t_2] \wedge [y_1, v] = [y_1, t_2] \quad \text{if } y_1 \leq t_2.$$

and $[u, t_2] \wedge [y_1, v] = \emptyset$ otherwise.

First we consider the case when $y_1 \not\leq t_2$. Then (4) yields

$$[u, x_2 \wedge t_2] = [u, t_2],$$

whence $t_2 \leq x_2$. The case $t_2 < x_2$ is impossible, since both t_2 and x_2 are covered by v . If $t_2 = x_2$, then $z = u$, which is a contradiction. Hence $y_1 \leq t_2$.

2.6. Lemma. *Assume that (α) is valid. Let $t \in P, t < v, t \neq x_2$. Then $y_1 \leq t$.*

Proof. Since $[u, t]$ is a dual atom in $\text{Int } P$, there is $z \in P$ such that $f(\{z\}) = [u, t]$.

From $t \neq x_2$ we infer that $z \neq u$. Therefore according to 2.5 the relation $y_1 \leq t$ is valid.

2.7. Lemma. *Assume that (α) and (β_1) hold. Let $z \neq v$. Then $t_1 \leq x_2$.*

Proof. By virtue of (3), the relation

- (5) $([u, x_2] \wedge [t_1, v]) \vee ([t_1, v] \wedge [y_1, v]) = [t_1, v]$

is valid. We have

$$\begin{aligned} [u, x_2] \wedge [t_1, v] &= [t_1, x_2] \quad \text{if } t_1 \leq x_2, \quad \text{and} \\ [u, x_2] \wedge [t_1, v] &= \emptyset \quad \text{otherwise.} \end{aligned}$$

Next, $[t_1, v] \wedge [y_1, v] = [t_1 \vee y_1, v]$.

If $t_1 \not\leq x_2$, then (5) implies that

$$[t_1 \vee y_1, v] = [t_1, v]$$

is valid, whence $y_1 \leq t_1$. The case $y_1 < t_1$ cannot occur, since $u < y_1$ and $u < t_1$. If $y_1 = t_1$, then $z = v$, which is a contradiction. Therefore $t_1 \leq x_2$.

2.8. Lemma. *Assume that (α) is valid. Let $t \in P$, $u < t$, $t \neq y_1$. Then $t \leq x_2$.*

The proof is analogous to that of 2.6 with the distinction that we apply 2.7 instead of 2.5.

2.9. Lemma. *Assume that (α) is valid. Let t be an element of P which does not belong to the set $\{u, v, x_2, y_1\}$. Then either $u < t < x_2$ or $y_1 < t < v$.*

Proof. In view of 2.4 we have either $u < t$ or $t < v$. Now it suffices to apply 2.6 and 2.8.

Under the assumption that (α) holds we denote

$$A = \{t \in P: u < t < x_2\}, \quad B = \{t \in P: y_1 < t < v\}.$$

2.10. Corollary. *Assume that (α) is valid. Then $A \cap B \neq \emptyset$.*

This is a consequence of 2.9 and of the fact that $\text{card } P > 4$.

The result of the above corollary can be sharpened by the following consideration.

2.11. Lemma. *Let (α) be valid and let $b \in B$. Then there is $a \in A$ such that $a < b$.*

Proof. In view of 2.2 there are dual atoms $[z_1, z_2]$ and $[z_3, z_4]$ of $\text{Int } P$ such that $[b, v] = [z_1, z_2] \wedge [z_3, z_4]$. Since $[b, v]$ is not a dual atom of $\text{Int } P$ we infer that $[z_1, z_2] \neq [z_3, z_4]$. Hence $z_1 = z_4 = v$ and $z_1 \neq z_3$. Next, z_1 and z_3 must belong to the set $A \cup \{y_1\}$. Thus either z_1 or z_3 belongs to A . Clearly $z_1 < b$ and $z_3 < b$.

2.12. Lemma. *Let (α) be valid and let $a \in A$. Then there is $b \in B$ such that $a < b$.*

The proof is analogous to that of 2.11.

2.13. Lemma. *Let (α) be valid. Then $A \neq \emptyset$ and $B \neq \emptyset$.*

Proof. This is a consequence of 2.10, 2.11 and 2.12.

2.14. Lemma. *The condition (α) cannot hold.*

Proof. By way of contradiction, suppose that (α) is valid. Then we have $\{u\} < [u, x_2]$, whence $f([u, x_2]) < f(\{u\}) = [u, x_2]$. Since $[u, x_2]$ is a dual atom of $\text{Int } P$, $f([u, x_2])$ must be an atom of $\text{Int } P$. Thus we have three possibilities:

- (a) $f([u, x_2]) = \{u\}$;
- (b) there is $a_1 \in A$ such that $f([u, x_2]) = \{a_1\}$;
- (c) $f([u, x_2]) = \{x_2\}$.

Next, the relation

$$(6) \quad f([u, x_2]) = f(\{u\} \vee \{x_2\}) = f(\{u\}) \wedge f(\{x_2\})$$

is valid.

First, suppose that (a) holds. Then in view of (6), $u \in f(\{x_2\})$. Because $f(\{x_2\})$ is a dual atom of $\text{Int } L$ and since $f(\{x_2\}) \neq f(\{u\}) = [u, x_2]$, there is $b_1 \in B$ such that $f(\{x_2\}) = [u, b_1]$. Thus (6) yields

$$\{u\} = [u, x_2] \wedge [u, b_1].$$

Hence no element of A is less than b_1 , contradicting 2.11.

Next, assume that (b) is valid. In view of (6) we infer

$$(7) \quad \{a_1\} = [u, x_2] \wedge f(\{x_2\}).$$

Thus $a_1 \in f(\{x_2\})$. Since $f(\{x_2\})$ is a dual atom of $\text{Int } L$ distinct from $[u, x_2]$, we have either

$$(8) \quad f(\{x_2\}) = [a_1, v],$$

or there is $b_1 \in B$ with $a_1 < b_1$ such that

$$(9) \quad f(\{x_2\}) = [u, b_1].$$

If (8) were valid we would have

$$[u, x_2] \wedge f(\{x_2\}) = [u, x_2] \wedge [a_1, v] \supset \{x_2\},$$

contradicting (7). If (9) holds, then $a \in [u, x_2] \wedge f(\{x_2\})$ and in view of (7) we arrive at a contradiction.

At last let us consider the case (c). Thus, according to (6),

$$(10) \quad \{x_2\} = [u, x_2] \wedge f(\{x_2\}).$$

Therefore $x_2 \in f(\{x_2\}) \neq [u, x_2]$. Since $f(\{x_2\})$ is a dual atom of $\text{Int } L$, there exists $a_1 \in A$ such that $f(\{x_2\}) = [a_1, v]$. Then

$$[u, x_2] \wedge f(\{x_2\}) = [u, x_2] \wedge [a_1, v] \supset \{a_1\};$$

in view of (10) we arrive at a contradiction.

2.15. Lemma. *The condition (β) cannot hold.*

The proof requires steps analogous to those which were applied in 2.5.–2.14. The details are omitted.

In view of 2.14 and 2.15 the proof of (B) is complete.

The following assertion is obvious.

2.16. Lemma. *Let P be a partially ordered set having the least and the largest element, and let $\text{card } P \leq 4$. Then P is a lattice.*

Theorems (A), (B) and Lemmas 2.1, 2.16 yield:

(C) Let P be a partially ordered set. Then the following conditions are equivalent:

- (i) The partially ordered set $\text{Int } P$ is selfdual.
- (ii) P is a lattice such that either $\text{card } P \leq 2$, or $\text{card } P = 4$ and P has two atoms.

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