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RELATION PRODUCTS OF CONGRUENCES AND FACTOR
CONGRUENCES

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Let A_1, A_2 be algebras of the same type. Denote by π_i the canonical projection from the Cartesian product $A_1 \times A_2$ onto A_i , $i = 1, 2$. Then $\Pi_i = \text{Ker } \pi_i$, $i = 1, 2$, are called *factor congruences* on $A_1 \times A_2$. A congruence Θ on $A_1 \times A_2$ is called a *subfactor congruence* whenever $\Theta \subseteq \Pi_1$ or $\Theta \subseteq \Pi_2$ hold.

Factor congruences and subfactor congruences were introduced by J. Hagemann [5]; see also H.-P. Gumm [4] for further information. The present paper studies four congruence properties on the product $A_1 \times A_2$ which are related to permutability and "3 1/2-permutability" of congruences on $A_1 \times A_2$ with factor congruences Π_1 and Π_2 . For the sake of brevity, denote the sequence p_1, \dots, p_m by \mathbf{p} .

Definition 1. Let A_1, A_2 be algebras of the same type. We say that the projection π_1 *preserves congruences* on $A_1 \times A_2$ whenever $\pi_1 \times \pi_1(\Theta) = \{ \langle a_1, a'_1 \rangle \in A_1 \times A_1; \langle a_1, a_2 \rangle \Theta \langle a'_1, a'_2 \rangle \text{ for some elements } a_2, a'_2 \in A_2 \}$ is a congruence on A_1 for any congruence Θ on $A_1 \times A_2$.

The property π_2 *preserves congruences* on $A_1 \times A_2$ is introduced analogously.

Theorem 1. For a variety \mathcal{V} , the following conditions are equivalent:

- (1) projections π_i , $i = 1, 2$, preserve congruences on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;
- (2) $\Theta \circ \Pi_1 \circ \Theta \subseteq \Pi_1 \circ \Theta \circ \Pi_1$ holds for any congruence Θ on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;
- (3) $\Pi_1 \circ \Theta \circ \Pi_1 = \Pi_1 \circ \Theta \circ \Pi_1 \circ \Theta = \Theta \circ \Pi_1 \circ \Theta \circ \Pi_1$ hold for any congruence Θ on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;
- (4) there exist ternary terms p_1, \dots, p_m , quaternary terms q_1, \dots, q_m , and $(4 + m)$ -ary terms s_1, \dots, s_n such that

$$\begin{aligned}
 & \begin{cases} x = s_1(x, y, y, z, \mathbf{p}(x, y, z)), \\ (\alpha) \begin{cases} s_i(y, x, z, y, \mathbf{p}(x, y, z)) = s_{i+1}(x, y, y, z, \mathbf{p}(x, y, z)), & 1 \leq i < n, \\ z = s_n(y, x, z, y, \mathbf{p}(x, y, z)), \end{cases} \\ (\beta) \begin{cases} s_i(y, x, u, z, \mathbf{q}(x, y, z, u)) = s_{i+1}(x, y, z, u, \mathbf{q}(x, y, z, u)), \\ 1 \leq i \leq n, \end{cases} \end{cases}
 \end{aligned}$$

are identities in \mathcal{V} .

Proof. (1) \Rightarrow (2): Take an arbitrary congruence Θ on the product $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$. Let $\langle x_1, x_2 \rangle \Theta \circ \Pi_1 \circ \Theta \langle y_1, y_2 \rangle$. Then $\langle x_1, x_2 \rangle \Theta \langle z_1, z_2 \rangle \Pi_1 \langle z_1, z_2 \rangle \Theta \langle y_1, y_2 \rangle$ for some elements $z_1 \in A_1$, $z_2, z'_2 \in A_2$. Since $\langle x_1, z_1 \rangle, \langle z_1, y_1 \rangle \in \pi_1 \times \pi_1(\Theta)$ we have $\langle x_1, y_1 \rangle \in \pi_1 \times \pi_1(\Theta)$, by hypothesis. This means that $\langle x_1, x_2 \rangle \Theta \langle y_1, y'_2 \rangle$ holds for some $x'_2, y'_2 \in A_2$. Altogether we find that $\langle x_1, x_2 \rangle \Pi_1 \langle x_1, x'_2 \rangle \Theta \langle y_1, y'_2 \rangle \Pi_1 \langle y_1, y_2 \rangle$ which implies $\langle x_1, x_2 \rangle \Pi_1 \circ \Theta \circ \Pi_1 \langle y_1, y_2 \rangle$. The inclusion $\Theta \circ \Pi_1 \circ \Theta \subseteq \Pi_1 \circ \Theta \circ \Pi_1$ is verified.

(2) \Rightarrow (3): The assumed inclusion $\Theta \circ \Pi_1 \circ \Theta \subseteq \Pi_1 \circ \Theta \circ \Pi_1$ yields $\Pi_1 \circ \Theta \circ \Pi_1 \circ \Theta \subseteq \Pi_1 \circ \Pi_1 \circ \Theta \circ \Pi_1 = \Pi_1 \circ \Theta \circ \Pi_1$. Further, $\Theta \circ \Pi_1 \circ \Theta \circ \Pi_1 = (\Pi_1 \circ \Theta \circ \Pi_1 \circ \Theta)^{-1} \subseteq (\Pi_1 \circ \Theta \circ \Pi_1)^{-1} = \Pi_1 \circ \Theta \circ \Pi_1$.

(3) \Rightarrow (2) is trivial.

(2) \Rightarrow (4): Take $A_1 = F_{\mathcal{V}}(x, y, z)$, $A_2 = F_{\mathcal{V}}(x, y, z, u)$, the \mathcal{V} -free algebras over three and four free generators, respectively. Let $\Theta = \Theta(\langle\langle x, x \rangle, \langle y, y \rangle\rangle, \langle\langle y, z \rangle, \langle z, u \rangle\rangle)$. Then $\langle x, x \rangle \Theta \circ \Pi_1 \circ \Theta \langle z, u \rangle$ since $\langle x, x \rangle \Theta \langle y, y \rangle \Pi_1 \langle y, z \rangle \Theta \langle z, u \rangle$. In virtue of hypothesis (2) we get that $\langle x, x \rangle \Pi_1 \circ \Theta \circ \Pi_1 \langle z, u \rangle$, which means that $\langle x, x \rangle \Pi_1 \langle x, c \rangle \Theta \langle z, d \rangle \Pi_1 \langle z, u \rangle$ for some elements $c, d \in A_2$. Applying the standard argument, see e.g. [2], to the relation $\langle\langle x, c \rangle, \langle z, d \rangle\rangle \in \Theta(\langle\langle x, x \rangle, \langle y, y \rangle\rangle, \langle\langle y, z \rangle, \langle z, u \rangle\rangle)$ we obtain the identities (4).

(4) \Rightarrow (1): Let Ψ be a congruence on the product $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$, and let $\langle a, b \rangle \Psi \langle c, d \rangle, \langle c, g \rangle \Psi \langle h, k \rangle$. Substitute $x = a, y = c, z = h$ in the identities (3) (α) and $x = b, y = d, z = g, u = k$ in the identities (3) (β) . Then

$$(\alpha) \quad a = s_1(a, c, c, h, \mathbf{p}(a, c, h)),$$

$$(\beta) \quad s_1 = s_1(b, d, g, k, \mathbf{q}(b, d, g, k)),$$

$$(\alpha) \quad s_i(c, a, h, c, \mathbf{p}(a, c, h)) = s_{i+1}(a, c, c, h, \mathbf{p}(a, c, h)),$$

$$(\beta) \quad s_i(d, b, k, g, \mathbf{q}(b, d, g, k)) = s_{i+1}(b, d, g, k, \mathbf{q}(b, d, g, k)), \quad 1 \leq i < n,$$

$$(\alpha) \quad h = s_n(c, a, h, c, \mathbf{p}(a, c, h)),$$

$$(\beta) \quad s_n = s_n(d, b, k, g, \mathbf{q}(b, d, g, k)),$$

from which the required conclusion $\langle a, s_1 \rangle \Psi \langle h, s_n \rangle$ readily follows. The proof is complete.

Remark 1. It is evident that conditions (2), (3) from Theorem 1 can be replaced by

(2') $\Theta \circ \Pi_2 \circ \Theta \subseteq \Pi_2 \circ \Theta \circ \Pi_2$ holds for any congruence Θ on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;

(3') $\Pi_2 \circ \Theta \circ \Pi_2 = \Pi_2 \circ \Theta \circ \Pi_2 \circ \Theta = \Theta \circ \Pi_2 \circ \Theta \circ \Pi_2$ hold for any congruence Θ on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$.

Theorem 2. For a variety \mathcal{V} , the following conditions are equivalent:

(1) projections π_i , $i = 1, 2$, preserve subfactor congruences on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;

(2) $\Theta_2 \circ \Pi_1 \circ \Theta_2 \subseteq \Pi_1 \circ \Theta_2 \circ \Pi_1$ holds for any subfactor congruence $\Theta_2 \subseteq \Pi_2$ on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;

(3) $\Pi_1 \circ \Theta_2 \circ \Pi_1 = \Pi_1 \circ \Theta_2 \circ \Pi_1 \circ \Theta_2 = \Theta_2 \circ \Pi_1 \circ \Theta_2 \circ \Pi_1$ hold for any subfactor congruence $\Theta_2 \subseteq \Pi_2$ on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;

(4) there exist ternary terms p_1, \dots, p_m , binary terms q_1, \dots, q_m , and $(4 + m)$ -ary terms t_1, \dots, t_n such that

$$\begin{aligned} (\alpha) \quad & \begin{cases} x = t_1(x, y, y, z, \mathbf{p}(x, y, z)), \\ t_i(y, x, z, y, \mathbf{p}(x, y, z)) = t_{i+1}(x, y, y, z, \mathbf{p}(x, y, z)), \quad 1 \leq i < n, \\ z = t_n(y, x, z, y, \mathbf{p}(x, y, z)), \end{cases} \\ (\beta) \quad & t_i(x, x, y, y, \mathbf{q}(x, y)) = t_{i+1}(x, x, y, y, \mathbf{q}(x, y)), \quad 1 \leq i \leq n, \end{aligned}$$

are identities in \mathcal{V} .

Proof. We omit the proof of part (1) \Rightarrow (2) \Leftrightarrow (3) as it runs in the same way as that of the foregoing Theorem 1.

(2) \Rightarrow (4): Choose $A_1 = F_{\mathcal{V}}(x, y, z)$, $A_2 = F_{\mathcal{V}}(x, y)$ and $\Theta = \Theta(\langle\langle x, x \rangle, \langle y, x \rangle\rangle, \langle\langle y, y \rangle, \langle z, y \rangle\rangle)$. Apparently Θ is a subfactor congruence on $A_1 \times A_2$, in particular $\Theta \subseteq \Pi_2$. Since $\langle x, x \rangle \Theta \langle y, x \rangle \Pi_1 \langle y, y \rangle \Theta \langle z, y \rangle$ the hypothesis (2) yields that $\langle x, x \rangle \Pi_1 \langle x, c \rangle \Theta \langle z, c \rangle \Pi_1 \langle z, y \rangle$ for an element $c \in A_2$. Applying [2] to the relation $\langle\langle x, c \rangle, \langle z, c \rangle\rangle \in \Theta(\langle\langle x, x \rangle, \langle y, x \rangle\rangle, \langle\langle y, y \rangle, \langle z, y \rangle\rangle)$ we immediately find the required identities (4).

(4) \Rightarrow (1): Let $\Psi \subseteq \Pi_2$ be a subfactor congruence on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$. Assume that $\langle a, b \rangle \Psi \langle c, b \rangle$, $\langle c, g \rangle \Psi \langle h, g \rangle$. Setting $x = a$, $y = c$, $z = h$ in the identities (4) (α) and $x = b$, $y = g$ in the remaining identities (4) (β) we obtain

$$\begin{aligned} (\alpha) \quad & a = t_1(a, c, c, h, \mathbf{p}(a, c, h)), \\ (\beta) \quad & t_1 = t_1(b, b, g, g, \mathbf{q}(b, g)), \\ (\alpha) \quad & t_i(c, a, h, c, \mathbf{p}(a, c, h)) = t_{i+1}(a, c, c, h, \mathbf{p}(a, c, h)), \\ (\beta) \quad & t_i(b, b, g, g, \mathbf{q}(b, g)) = t_{i+1}(b, b, g, g, \mathbf{q}(b, g)), \quad 1 \leq i < n, \\ (\alpha) \quad & h = t_n(c, a, h, c, \mathbf{p}(a, c, h)), \\ (\beta) \quad & t_n = t_n(b, b, g, g, \mathbf{q}(b, g)). \end{aligned}$$

In this way we obtain $\langle a, t_1 \rangle \Psi \langle h, t_1 \rangle$ which establishes the transitivity of $\pi_1 \times \pi_1(\Psi)$. The proof is complete.

Let R be a binary relation on A , $a \in A$. Then $[a]R$ denotes the subset $\{x \in A; \langle a, x \rangle \in R\}$ of A .

Definition 2. Let A_1, A_2 be algebras of the same type. We say that the projection π_1 preserves blocks of congruences on $A_1 \times A_2$ whenever $\pi_1([\langle a_1, a_2 \rangle] \Theta) = [\pi_1(\langle a_1, a_2 \rangle)] \pi_1 \times \pi_1(\Theta)$ holds for any elements $a_i \in A_i$, $i = 1, 2$, and any congruence Θ on $A_1 \times A_2$.

The property π_2 preserves blocks of congruences on $A_1 \times A_2$ is introduced analogously.

Theorem 3. Let A_1, A_2 be algebras of the same type. The following conditions are equivalent:

- (1) projection π_1 preserves blocks of congruences on $A_1 \times A_2$;
(2) $\Pi_1 \circ \Theta = \Theta \circ \Pi_1$ holds for any congruence Θ on $A_1 \times A_2$.

Proof. (1) \Rightarrow (2): Let $\langle x_1, x_2 \rangle \Pi_1 \circ \Theta \langle y_1, y_2 \rangle$. Then $\langle x_1, x_2 \rangle \Pi_1 \langle x_1, x'_2 \rangle \Theta \langle y_1, y_2 \rangle$ for an element $x'_2 \in A_2$. In particular we have $\langle x_1, y_1 \rangle \in \pi_1 \times \pi_1(\Theta)$, which can be expressed as $\langle \pi_1(\langle x_1, x_2 \rangle), \pi_1(\langle y_1, y_2 \rangle) \rangle \in \pi_1 \times \pi_1(\Theta)$ or, equivalently, $\pi_1(\langle y_1, y_2 \rangle) \in [\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta)$. By hypothesis $[\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta) = \pi_1([\langle x_1, x_2 \rangle] \Theta)$ and so $\pi_1(\langle y_1, y_2 \rangle) \in \pi_1([\langle x_1, x_2 \rangle] \Theta)$. Hence $\langle y_1, y'_2 \rangle \Theta \langle x_1, x_2 \rangle$ for some $y'_2 \in A_2$. In this way we get $\langle x_1, x_2 \rangle \Theta \langle y_1, y'_2 \rangle \Pi_1 \langle y_1, y_2 \rangle$, which proves the inclusion $\Pi_1 \circ \Theta \subseteq \Theta \circ \Pi_1$. Consequently also $\Theta \circ \Pi_1 = (\Pi_1 \circ \Theta)^{-1} \subseteq (\Theta \circ \Pi_1)^{-1} = \Pi_1 \circ \Theta$. Altogether $\Pi_1 \circ \Theta = \Theta \circ \Pi_1$, as required.

(2) \Rightarrow (1): We want to verify the equality $\pi_1([\langle x_1, x_2 \rangle] \Theta) = [\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta)$ for any elements $x_i \in A_i$, $i = 1, 2$, and any congruence Θ on $A_1 \times A_2$.

(i) The inclusion $\pi_1([\langle x_1, x_2 \rangle] \Theta) \subseteq [\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta)$ is trivial.

(ii) Conversely, let $y_1 \in [\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta) = [x_1] \pi_1 \times \pi_1(\Theta)$. Equivalently $\langle x_1, y_1 \rangle \in \pi_1 \times \pi_1(\Theta)$, which means that $\langle y_1, y_2 \rangle \Theta \langle x_1, x'_2 \rangle$ for some elements $y_2, x'_2 \in A_2$. Since $\langle y_1, y_2 \rangle \Theta \langle x_1, x'_2 \rangle \Pi_1 \langle x_1, x_2 \rangle$ we get $\langle y_1, y_2 \rangle \Pi_1 \langle y_1, y'_2 \rangle \Theta \langle x_1, x_2 \rangle$ for an element $y'_2 \in A_2$, by hypothesis. In other words $\langle y_1, y'_2 \rangle \in [\langle x_1, x_2 \rangle] \Theta$ and so $y_1 = \pi_1(\langle y_1, y'_2 \rangle) \in \pi_1([\langle x_1, x_2 \rangle] \Theta)$. The proof is complete.

Remark 2. The equivalent conditions from the foregoing Theorem 3 defined the so called *factor permutable varieties* (briefly: FP-varieties). Mal'cev characterizations of FP-varieties were given by J. Hagemann [5] and by H.-P. Gumm [4].

Definition 3. Let A_1, A_2 be algebras of the same type. A congruence Θ on $A_1 \times A_2$ is called *factorable* whenever $\Theta = \Theta_1 \times \Theta_2 = \{ \langle \langle a_1, a_2 \rangle, \langle a'_1, a'_2 \rangle \rangle; a_i \Theta_i a'_i, i = 1, 2 \}$ for some congruences Θ_i on A_i , $i = 1, 2$.

A variety \mathcal{V} has *factorable subfactor congruences* whenever any subfactor congruence on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$, has this property.

Theorem 4. For a variety \mathcal{V} , the following conditions are equivalent:

- (1) projections π_i , $i = 1, 2$, preserve blocks of subfactor congruences on $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;
(2) $\Theta_1 \circ \Pi_2 = \Pi_2 \circ \Theta_1$ holds for any subfactor congruence $\Theta_1 \subseteq \Pi_1$ on the product $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;
(3) $\Theta_1 \circ \Theta_2 = \Theta_2 \circ \Theta_1$ holds for any subfactor congruences $\Theta_i \subseteq \Pi_i$, $i = 1, 2$, on the product $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$;
(4) \mathcal{V} has factorable subfactor congruences.

Proof: (1) \Leftrightarrow (2): See the proof of Theorem 3.

(3) \Rightarrow (2) is trivial.

(2) \Rightarrow (4): Consider the principal subfactor congruence $\Theta_1 = \Theta(\langle \langle x, x \rangle, \langle x, y \rangle \rangle) \subseteq \Pi_1$ on the product $A_1 \times A_2 = F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$. Since $\langle x, y \rangle \Pi_2$

$\langle y, y \rangle$ holds evidently we get $\langle x, x \rangle \Theta_1 \circ \Pi_2 \langle y, y \rangle$. By hypothesis (2) also $\langle x, x \rangle \Pi_2 \circ \Theta_1 \langle y, y \rangle$, which means that $\langle x, x \rangle \Pi_2 \langle s_1, s_2 \rangle \Theta_1 \langle y, y \rangle$ for some elements $s_i \in A_i$, $i = 1, 2$. Then $s_1 = y$, $s_2 = x$ and so $\langle \langle y, x \rangle, \langle y, y \rangle \rangle \in \Theta_1 = \Theta(\langle \langle x, x \rangle, \langle x, y \rangle \rangle)$. The last condition implies condition (4), as was already shown by J. Hagemann [5] and by I. Chajda [1].

(4) \Rightarrow (3): Let $\Theta_i \subseteq \Pi_i$, $i = 1, 2$, be arbitrary subfactor congruences on the product $A_1 \times A_2$, $A_i \in \mathcal{V}$, $i = 1, 2$. Take $\langle x_1, x_2 \rangle \Theta_1 \circ \Theta_2 \langle y_1, y_2 \rangle$. Then $\langle x_1, x_2 \rangle \Theta_1 \langle s_1, s_2 \rangle \Theta_2 \langle y_1, y_2 \rangle$ for some elements $s_i \in A_i$, $i = 1, 2$. Since Θ_1, Θ_2 are subfactor congruences we have $s_1 = x_1$ and $s_2 = y_2$. By hypothesis (4) $\langle x_1, x_2 \rangle \Theta_1 \langle x_1, y_2 \rangle$ implies $\langle y_1, x_2 \rangle \Theta_1 \langle y_1, y_2 \rangle$ and $\langle x_1, y_2 \rangle \Theta_2 \langle y_1, y_2 \rangle$ implies $\langle x_1, x_2 \rangle \Theta_2 \langle y_1, x_2 \rangle$. Altogether we have $\langle x_1, x_2 \rangle \Theta_2 \langle y_1, x_2 \rangle \Theta_1 \langle y_1, y_2 \rangle$, i.e. $\langle x_1, x_2 \rangle \Theta_2 \circ \Theta_1 \langle y_1, y_2 \rangle$, which establishes the permutability of subfactor congruences Θ_1, Θ_2 . The proof is complete.

Remark 3. Mal'cev conditions for varieties with factorable subfactor congruences can be found in J. Hagemann [5] and also in I. Chajda [1].

Examples. Any variety with permutable congruences as well as any variety with factorable congruences evidently have all properties listed in the above Theorem 1, 2, 3, 4.

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