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On the convergence of Neumann series for noncompact operators


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General geometric conditions on an open set $G \subset \mathbb{R}^m$ with a compact boundary $\partial G$ are known which permit to represent the solution of the Dirichlet problem with a prescribed boundary condition $g \in C(\partial G)$ by means of double layer potential with a continuous momentum density $f \in C(\partial G)$. This problem reduces to the equation 

\[(I + T)f = h\]

for the unknown $f \in C(\partial G)$, where $h = 2g$ and $T$ is the Neumann operator of the arithmetical mean acting on $C(\partial G)$. Similarly, the solution of the Neumann problem for the complementary domain, where the prescribed normal derivative on $\partial G$ is weakly characterized by a signed measure $\mu$, can be represented by a single layer potential of a signed measure $\nu$ satisfying 

\[(I + T')\nu = 2\mu,\]

where the dual operator $(I + T')'$ acts on the space $C'(\partial G)$ of all signed measure supported by $\partial G$ (cf. [K]). Historically the Neumann series occurred in connection with attempts to invert the operators $I + T$, $(I + T')'$ in the case when $G$ or its complement is convex the operator of the arithmetical mean, considered on the factorspace $C(\partial G)$ modulo the subspace of constant functions on $\partial G$, has the spectral radius less than 1. Further development led to the Riesz-Schauder theory of the dual equations (1), (2) for the case that $T$ is a compact linear operator acting on a Banach space $X$. It was shown much later in [S] that in this case the Neumann series 

\[
\sum_{n=0}^{\infty} (-1)^n T^n h
\]

converges to a solution $f \in X$ of the equation (1) if and only if the sequence $T^n h$ tends to zero in $X$ as $n \to \infty$. Unfortunately, potential-theoretic boundary value problems lead to equations (1), (2) with a compact $T$ only if the boundary $\partial G$ is sufficiently smooth. As observed already by J. Radon ([R]), in order to allow non-smooth boundaries it is useful to consider the equations (1), (2) for more general operators $T$ such that $\omega(T) < 1$, where $\omega(T)$ denotes the distance of $T$ from the 

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subspace of all compact linear operators. (For the Neumann operator $T$ of the arithmetical mean, $\omega(T)$ can be evaluated in geometric terms in dependence on the structure of $\partial G$; simple examples in [AKK], [KW] show that it is often useful to introduce a new norm in $C(\partial G)$ inducing the same topology of uniform convergence in order to achieve $\omega(T) < 1$.)

It is the aim of the present paper to show that the results established in [S] for compact $T$ remain in force if $\omega(T) < 1$.

**Lemma 1.** Let $X$ be a Banach space, let $U, K$ be bounded linear operators on $X$, $K$ compact, $\|U\| < 1/2$. Denote by $\sigma(U + K)$ the spectrum of the operator $K + U$. Then there exists $d \in (0, 1)$ such that $\sigma(K + U) \cap \{\lambda; |\lambda| > d\}$ is a finite set.

**Proof.** Denote $r = \|U\|$. Choose $d \in (2r, 1)$, $p \in (2rd, 1)$. Suppose that there exists a simple sequence $\{\lambda_i\} \subset \sigma(K + U) \cap \{\lambda; |\lambda| > d\}$. For every natural number $i, \lambda_i$ does not lie in the essential spectrum of the operator $(U + K)$ and according to [Sch], Chapter 7, Theorem 5.4 $(\lambda_i I - U - K)$ is a Fredholm operator with index 0 (where $I$ is the identical operator) and thus $\lambda_i$ is an eigenvalue of the operator $(U + K)$. The null spaces $N(\lambda_i I - U - K)$ of the operators $(\lambda_i I - U - K)$ have finite dimensions and therefore they are closed subspaces of $X$. Denote by $X_n$ the direct sum of the spaces $N(\lambda_1 I - U - K), \ldots, N(\lambda_n I - U - K)$. Since $X_n \neq X_{n+1}$, there exist unit vectors $y_{n+1} \in X_{n+1}$ such that $\text{dist}(y_{n+1}, X_n) > p$ in view of the Riesz lemma (see [T], Theorem 3.12-E). Since for $y_{n+1}$ there exist $x_i \in N(\lambda_i I - K - U), i = 1, \ldots, n + 1$, such that

$$y_{n+1} = \sum_{i=1}^{n+1} x_i,$$

we have

$$(\lambda_{n+1} I - U - K) y_{n+1} = \sum_{i=1}^{n+1} (\lambda_{n+1} - \lambda_i) x_i \in X_n.$$  

If $n > m$ then

$$\left\| (K + U) \frac{1}{\lambda_n} y_n - (K + U) \frac{1}{\lambda_m} y_m \right\| =$$

$$= \left\| y_n - \left[ y_m - \frac{1}{\lambda_m} (\lambda_m I - U - K) y_m + \frac{1}{\lambda_n} (\lambda_n I - U - K) y_n \right] \right\| > p,$$

because $[y_m - (1/\lambda_m) (\lambda_m I - U - K) y_m + (1/\lambda_n) (\lambda_n I - U - K) y_n] \in X_{n-1}$. Thus

$$\left\| K \left( \frac{1}{\lambda_n} y_n \right) - K \left( \frac{1}{\lambda_m} y_m \right) \right\| \geq \left\| (K + U) \frac{1}{\lambda_n} y_n - (K + U) \frac{1}{\lambda_m} y_m \right\| -$$

$$- \left\| U \left( \frac{1}{\lambda_n} y_n - \frac{1}{\lambda_m} y_m \right) \right\| > p - \frac{2r}{d},$$

which contradicts compactness of $K$.  

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Lemma 2. Let $X$ be a complex Banach space, let $U, K$ be bounded linear operators on $X$, $K$ compact, $\|U\| < 1$. Then there is $d \in (0, 1)$ such that the set $\sigma(K + U) \cap \{\lambda; |\lambda| > d\}$ is finite.

Proof. Since $\|U\| < 1$ there exists a natural number $n$ such that $\|U\| < 1/2$. Since $(U + K)^n = U^n + L$, where $L$ is a compact operator on $X$, by virtue of Lemma 1 there is a number $d \in (0, 1)$ such that $\sigma((K + U)^n) \cap \{\lambda; |\lambda| > d\}$ is finite. Since $\sigma((K + U)^n) = \{\lambda^n; \lambda \in \sigma(K + U)\}$ according to [Sch], Chapter 6, Theorem 3.8, the set $\sigma(K + U) \cap \{\lambda; |\lambda| > \sqrt[d]{d}\}$ is finite.

Theorem. Let $X$ be a Banach space, let $U, K$ be bounded linear operators on $X$ such that $K$ is compact and $\|U\| < 1$. If $x \in X$, then the series $\sum_{n=0}^{\infty} (U + K)^n x$ converges if and only if $(U + K)^n x \to 0$ as $n \to \infty$.

Proof. It suffices to prove that $(U + K)^n x \to 0$ implies that the series $\sum_{n=0}^{\infty} (U + K)^n x$ converges. Denote $A = U + K$. If $X$ is a real Banach space denote by $\bar{X} = \{[z_1, z_2]; z_1, z_2 \in X\}$ the complex Banach space for which $[z_1, z_2] + [y_1, y_2] = [z_1 + y_1, z_2 + y_2], (a_1 + i a_2) [z_1, z_2] = [a_1 z_1 - a_2 z_2, a_1 z_2 + a_2 z_1]$, $\| [z_1, z_2] \| = \sqrt{\|z_1\|^2 + \|z_2\|^2}$. We embed the space $X$ into $\bar{X}$ in such a way that we identify $z$ and $[z, 0]$. If we define the linear operator $\bar{A}$ on $\bar{X}$ by $\bar{A}[z_1, z_2] = [A z_1, A z_2]$ we may confine ourselves to the case that $X$ is complex.

Let $X$ be a complex Banach space. By Lemma 2 there is natural number $n$ such that $\sigma(U + K) = \{\lambda_1, \ldots, \lambda_{n-1}\} \cup (\sigma(K + U) \cap \{\lambda; |\lambda| < 1\})$ and if we denote $\sigma_n = \sigma(U + K) \cap \{\lambda; |\lambda| < 1\}, \sigma_i = \{\lambda_i\}$ for $i = 1, \ldots, n - 1$, the sets $\sigma_i$ are disjoint and closed. Choose disjoint open sets $V_1, \ldots, V_n$ in the complex plane such that $\sigma_i \subset V_i$ for $i = 1, \ldots, n$. For $i \in \{1, \ldots, n\}$ we define on $\bigcup \{V_j; j = 1, \ldots, n\}$ functions

$$f_i(y) = 1 \quad \text{for} \quad y \in V_i,$$

$$= 0 \quad \text{for} \quad y \notin V_i.$$

Then $f_i(A)$ are bounded projections on $X$ such that $f_1(A) + \ldots + f_n(A) = I$, where $I$ is the identical operator and $A$ maps $f_i(A)(X)$ into $f_i(A)(X)$ (see [Sch], Chapter 6). We prove that $f_i(A) x = 0$ for $i = 1, \ldots, n - 1$. Since

$$A^m f_1(A) x + \ldots + A^m f_n(A) x = A^m x = f_1(A) A^m x + \ldots + f_n(A) A^m x$$

and the space $X$ is the direct sum of the subsets $f_1(A)(X), \ldots, f_n(A)(X)$, we have $A^m f_i(A) x = f_i(A) A^m x \to 0$ as $m \to \infty$ for $i \in \{1, \ldots, n\}$. Denote by $A_i$ the restriction of the operator $A$ to the space $f_i(A)(X)$ $(i = 1, \ldots, n)$. According to [Sch], Chapter 6, Theorem 4.1, $\sigma(A_i) = \sigma_i$ for $i = 1, \ldots, n$.

Now fix $i \in \{1, \ldots, n - 1\}$. Since $\lambda_i$ does not lie in the essential spectrum of the operator $A$ because $\|U\| < 1$, the operator $(\lambda_i I - A)$ is a Fredholm operator with index 0 according to [Sch], Chapter 7, Theorem 5.4. Since the space $X$ is the direct sum of the subspaces $f_1(A)(X), \ldots, f_n(A)(X)$, the subspace $(\lambda_i I - A)(X)$ is the direct sum of the subspaces $(\lambda_i I - A)(f_1(X)), \ldots, (\lambda_i I - A_n)(f_n(X))$. Since
codim \((\lambda I - A)(X)\) < \(\infty\), we have \(\text{codim} (\lambda_1 I - A_1)(f_1(X)) < \infty\). At the same time \((\lambda I - A_1)(f_1(X)) = (\lambda I - A)(X) \cap f_1(X)\) is a closed subspace of \(f_1(X)\). Since the dimension of the null space of the operator \((\lambda I - A_1)\) is less than or equal to the dimension of the null space of the operator \((\lambda I - A)\), the operator \((\lambda I - A_1)\) is Fredholm. Since \(\sigma(A_1) = \{\lambda_1\}\), the operator \(\lambda I - A_1\) is Fredholm for each complex number \(\lambda\). According to [Sch], Chapter 9, Theorem 2.2 the space \(f_1(A)(X)\) has a finite dimension. Since \(f_1(A)(X)\) is a finite dimensional space and \(\sigma(A_1) = \{\lambda_1\}\), and according to [H], § 58, Theorem 2 there is a natural number \(m\) such that \((\lambda I - A_1)^m = 0\). If \(f_1(A)x \neq 0\), then there is a natural number \(k\) such that \(v = (\lambda I - A_1)^{k-1} f_1(A)x = 0\). Since \(A^j f_1(A)x \to 0\) as \(j \to \infty\), we have \(A^{j+r} f_1(A)x \to 0\) for \(j \to \infty\) and every fixed natural number \(r\). Thus \(A^j v \to 0\) as \(j \to \infty\). But \(Av = \lambda_1 v\) and thus \(\|A^j v\| = |\lambda_1^j| \|v\| \geq \|v\|\), which is a contradiction. Hence \(f_1(A)x = 0\).

Therefore \(x \in f_1(A)(X)\). Since the spectral radius of the operator \(A_n\) is less than 1, the series \[\sum_{k=0}^{\infty} A^k x = \sum_{k=0}^{\infty} A_n^k x\]
converges.

**Note:** Let \(X\) be a Banach space. Suppose that \(U, K\) are bounded linear operators on \(X\) such that \(K\) is compact and \(\|U\| < 1\). If \(x \in X\) then the series \(\sum_{n=0}^{\infty} (U + K)^n x\) converges if and only if \((U + K)^n x\) converges weakly to zero as \(n \to \infty\).

**Proof.** According to Theorem it suffices to prove that if \((U + K)^n x\) converges weakly to zero then it converges to zero. Suppose the contrary. Then there exist \(\varepsilon > 0\) and a subsequence \(\{n_k\}\) such that \(\|(U + K)^k x\| > \varepsilon\) for each \(k\). Since \((U + K)^n x\) converges weakly to zero it is bounded according to [T], Theorem 4.4-D. There is a positive constant \(M\) such that
\[\|(U + K)^n x\| \leq M\]
for each natural \(n\). Since \(\|U\| < 1\) there exists a natural number \(n_0\) such that
\[\|U\|^{n_0} < \frac{\varepsilon}{4M}\]
According to [DSch], Chapter VI, § 5, Theorem 4 the operator \(L = (U + K)^{n_0} - U^{n_0}\) is compact. By virtue of (5) there is a subsequence \(\{m_j\}\) of \(\{n_k\}\) and \(y \in X\) such that \(L(U + K)^{m_j - n_0} x\) converges to \(y\). Since \((L + U^{n_0})(U + K)^{m_j - n_0} x\) converges weakly to zero and \(L(U + K)^{m_j - n_0} x\) converges to \(y\), the sequence \(U^{n_0}(U + K)^{m_j - n_0} x\) converges weakly to \((-y)\). Now (5), (6) imply
\[\|U^{n_0}(U + K)^{m_j - n_0} x\| < \frac{\varepsilon}{4}\]
If we consider \(y\) and \(U^{n_0}(U + K)^{m_j - n_0} x\) as elements of the second dual of \(X\) we obtain
\|y\| \leq \varepsilon/4. Since \(L(U + K)^{m_j-n_0} x\) converges to \(y\) there is \(m_j\) for which

\begin{equation}
\|L(U + K)^{m_j-n_0} x\| < \frac{\varepsilon}{2}.
\end{equation}

From (7), (8) we conclude

\[ \|(U + K)^{m_j} x\| < \frac{1}{2} \varepsilon, \]

which contradicts \(\|(U + K)^{m_j} x\| > \varepsilon.\)

References


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