Dagmar Medková

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DAGMAR MEDKOVÁ, Praha

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General geometric conditions on an open set $G \subset R^m$ with a compact boundary ∂G are known which permit to represent the solution of the Dirichlet problem with a prescribed boundary condition $g \in C(\partial G)$ by means of double layer potential with a continuous momentum density $f \in C(\partial G)$. This problem reduces to the equation

(1) $\qquad (I + T) f = h$

for the unknown $f \in C(\partial G)$, where $h = 2g$ and T is the Neumann operator of the arithmetical mean acting on $C(\partial G)$. Similarly, the solution of the Neumann problem for the complementary domain, where the prescribed normal derivative on ∂G is weakly characterized by a signed measure μ, can be represented by a single layer potential of a signed measure ν satisfying

(2) $\qquad (I + T)' \nu = 2\mu ,$

where the dual operator $(I + T)'$ acts on the space $C'(\partial G)$ of all signed measure supported by ∂G (cf. [Kl]). Historically the Neumann series occurred in connection with attempts to invert the operators $I + T$, $(I + T)'$ in the case when G or its complement is convex the operator of the arithmetical mean, considered on the factorspace $C(\partial G)$ modulo the subspace of constant functions on ∂G, has the spectral radius less than 1. Further development led to the Riesz-Schauder theory of the dual equations (1), (2) for the case that T is a compact linear operator acting on a Banach space X. It was shown much later in [S] that in this case the Neumann series

$$\sum_{n=0}^{\infty} (-1)^n T^n h$$

converges to a solution $f \in X$ of the equation (1) if and only if the sequence $T^n h$ tends to zero in X as $n \to \infty$. Unfortunately, potential-theoretic boundary value problems lead to equations (1), (2) with a compact T only if the boundary ∂G is sufficiently smooth. As observed already by J. Radon ([R]), in order to allow non-smooth boundaries it is useful to consider the equations (1), (2) for more general operators T such that $\omega(T) < 1$, where $\omega(T)$ denotes the distance of T from the subspace of all compact linear operators. (For the Neumann operator T of the arithmetical mean, $\omega(T)$ can be evaluated in geometric terms in dependence on the structure of ∂G; simple examples in [AKK], [KW] show that it is often useful to introduce a new norm in $C(\partial G)$ inducing the same topology of uniform convergence in order to achieve $\omega(T) < 1$.)

It is the aim of the present paper to show that the results established in [S] for compact T remain in force if $\omega(T) < 1$.

Lemma 1. *Let X be a Banach space, let U, K be bounded linear operators on X, K compact, $\|U\| < 1/2$. Denote by $\sigma(U + K)$ the spectrum of the operator $K + U$. Then there exists $d \in (0, 1)$ such that $\sigma(K + U) \cap \{\lambda; |\lambda| > d\}$ is a finite set.*

Proof. Denote $r = \|U\|$. Choose $d \in (2r, 1)$, $p \in (2r/d, 1)$. Suppose that there exists a simple sequence $\{\lambda_i\} \subset \sigma(K + U) \cap \{\lambda; |\lambda| > d\}$. For every natural number i, λ_i does not lie in the essential spectrum of the operator $(U + K)$ and accordig to [Sch], Chapter 7, Theorem 5.4 $(\lambda_i I - U - K)$ is a Fredholm operator with index 0 (where I is the identical operator) and thus λ_i is an eigenvalue of the operator $(U + K)$. The null spaces $N(\lambda_i I - U - K)$ of the operators $(\lambda_i I - U - K)$ have finite dimensions and therefore they are closed subspaces of X. Denote by X_n the direct sum of the spaces $N(\lambda_1 I - U - K), \ldots, N(\lambda_n I - U - K)$. Since $X_n \neq X_{n+1}$, there exist unit vectors $y_{n+1} \in X_{n+1}$ such that $\operatorname{dist}(y_{n+1}, X_n) > p$ in view of the Riesz lemma (see [T], Theorem 3.12-E). Since for y_{n+1} there exist $x_i \in N(\lambda_i I - K - U)$, $i = 1, \ldots, n + 1$, such that

$$y_{n+1} = \sum_{i=1}^{n+1} x_i ,$$

we have

$$(\lambda_{n+1} I - U - K)\, y_{n+1} = \sum_{i=1}^{n+1} (\lambda_{n+1} - \lambda_i)\, x_i \in X_n .$$

If $n > m$ then

$$\left\|(K + U)\frac{1}{\lambda_n}\, y_n - (K + U)\frac{1}{\lambda_m}\, y_m\right\| =$$

$$= \left\|y_n - \left[y_m - \frac{1}{\lambda_m}(\lambda_m I - U - K)\, y_m + \frac{1}{\lambda_n}(\lambda_n I - U - K)\, y_n\right]\right\| > p ,$$

because $[y_m - (1/\lambda_m)(\lambda_m I - U - K)\, y_m + (1/\lambda_n)(\lambda_n I - U - K)\, y_n] \in X_{n-1}$. Thus

$$\left\|K\left(\frac{1}{\lambda_n}\, y_n\right) - K\left(\frac{1}{\lambda_m}\, y_m\right)\right\| \geqq \left\|(K + U)\frac{1}{\lambda_n}\, y_n - (K + U)\frac{1}{\lambda_m}\, y_m\right\| -$$

$$- \left\|U\left(\frac{1}{\lambda_n}\, y_n - \frac{1}{\lambda_m}\, y_m\right)\right\| > p - \frac{2r}{d} ,$$

which contradicts compactness of K.

Lemma 2. *Let X be a complex Banach space, let U, K be bounded linear operators on X, K compact, $\|U\| < 1$. Then there is $d \in (0, 1)$ such that the set $\sigma(K + U) \cap \cap \{\lambda; |\lambda| > d\}$ is finite.*

Proof. Since $\|U\| < 1$ there exists a natural number n such that $\|U\|^n < 1/2$. Since $(U + K)^n = U^n + L$, where L is a compact operator on X, by virtue of Lemma 1 there is a number $d \in (0, 1)$ such that $\sigma((K + U)^n) \cap \{\lambda; |\lambda| > d\}$ is finite. Since $\sigma((K + U)^n) = \{\lambda^n; \lambda \in \sigma(K + U)\}$ according to [Sch], Chapter 6, Theorem 3.8, the set $\sigma(K + U) \cap \{\lambda; |\lambda| > \sqrt[n]{d}\}$ is finite.

Theorem. *Let X be a Banach space, let U, K be bounded linear operators on X such that K is compact and $\|U\| < 1$. If $x \in X$, then the series $\sum_{n=0}^{\infty} (U + K)^n x$ converges if and only if $(U + K)^n x \to 0$ as $n \to \infty$.*

Proof. It suffices to prove that $(U + K)^n x \to 0$ implies that the series $\sum (U + K)^n x$ converges. Denote $A = U + K$. If X is a real Banach space denote by $\tilde{X} = \{[z_1, z_2]; z_1, z_2 \in X\}$ the complex Banach space for which $[z_1, z_2] + [y_1, y_2] = [z_1 + y_1, z_2 + y_2], (\alpha_1 + i\alpha_2) [z_1, z_2] = [\alpha_1 z_1 - \alpha_2 z_2, \alpha_1 z_2 + \alpha_2 z_1]$, $\|[z_1, z_2]\| = \sqrt{(\|z_1\|^2 + \|z_2\|^2)}$. We embed the space X into $\tilde{X}$ in such a way that we identify z and $[z, 0]$. If we define the linear operator $\tilde{A}$ on $\tilde{X}$ by $\tilde{A}[z_1, z_2] = [Az_1, Az_2]$ we may confine ourselves to the case that X is complex.

Let X be a complex Banach space. By Lemma 2 there is natural number n such that $\sigma(U + K) = \{\lambda_1, \ldots, \lambda_{n-1}\} \cup (\sigma(K + U) \cap \{\lambda; |\lambda| < 1\})$ and if we denote $\sigma_n = \sigma(U + K) \cap \{\lambda; |\lambda| < 1\}$, $\sigma_i = \{\lambda_i\}$ for $i = 1, \ldots, n - 1$, the sets σ_i are disjoint and closed. Choose disjoint open sets $V_1, \ldots, V_n$ in the complex plane such that $\sigma_i \subset V_i$ for $i = 1, \ldots, n$. For $i \in \{1, \ldots, n\}$ we define on $\bigcup \{V_j; j = 1, \ldots, n\}$ functions

$$\begin{aligned} f_i(y) &= 1 \quad \text{for} \quad y \in V_i\,, \\ &= 0 \quad \text{for} \quad y \notin V_i\,. \end{aligned}$$

Then $f_i(A)$ are bounded projections on X such that $f_1(A) + \ldots + f_n(A) = I$, where I is the identical operator and A maps $f_i(A)(X)$ into $f_i(A)(X)$ (see [Sch], Chapter 6). We prove that $f_i(A)\,x = 0$ for $i = 1, \ldots, n - 1$. Since

$$A^m f_1(A)\,x + \ldots + A^m f_n(A)\,x = A^m x = f_1(A)\,A^m x + \ldots + f_n(A)\,A^m x$$

and the space X is the direct sum of the subsets $f_1(A)(X), \ldots, f_n(A)(X)$, we have $A^m f_i(A)\,x = f_i(A)\,A^m x \to 0$ as $m \to \infty$ for $i \in \{1, \ldots, n\}$. Denote by A_i the restriction of the operator A to the space $f_i(A)(X)$ $(i = 1, \ldots, n)$. According to [Sch], Chapter 6, Theorem 4.1, $\sigma(A_i) = \sigma_i$ for $i = 1, \ldots, n$.

Now fix $i \in \{[1, \ldots, n - 1]\}$. Since λ_i does not lie in the essential spectrum of the operator A because $\|U\| < 1$, the operator $(\lambda_i I - A)$ is a Fredholm operator with index 0 according to [Sch], Chapter 7, Theorem 5.4. Since the space X is the direct sum of the subspaces $f_1(A)(X), \ldots, f_n(A)(X)$, the subspace $(\lambda_i I - A)(X)$ is the direct sum of the subspaces $(\lambda_i I - A_1)(f_1(X)), \ldots, (\lambda_i I - A_n)(f_n(X))$. Since codim $(\lambda_i I - A)(X) < \infty$, we have codim $(\lambda_i I - A_i)(f_i(X)) < \infty$. At the same time $(\lambda_i I - A_i)(f_i(X)) = (\lambda_i I - A)(X) \cap f_i(X)$ is a closed subspace of $f_i(X)$. Since the dimension of the null space of the operator $(\lambda_i I - A_i)$ is less than or equal to the dimension of the null space of the operator $(\lambda_i I - A)$, the operator $(\lambda_i I - A_i)$ is Fredholm. Since $\sigma(A_i) = \{\lambda_i\}$, the operator $\lambda I - A_i$ is Fredholm for each complex number λ. According to [Sch], Chapter 9, Theorem 2.2 the space $f_i(A)(X)$ has a finite dimension. Since $f_i(A)(X)$ is a finite dimensional space and $\sigma(A_i) = \{\lambda_i\}$ and according to [H], § 58, Theorem 2 there is a natural number m such that $(\lambda_i I - A_i)^m = 0$. If $f_i(A)\, x \neq 0$, then there is a natural number k such that $v = (\lambda_i I - A_i)^{k-1} \, . \, f_i(A)\, x \neq 0$, $(\lambda_i I - A_i)^k f_i(A)\, x = 0$. Since $A^j f_i(A)\, x \to 0$ as $j \to \infty$, we have $A^{j+r} f_i(A)\, x \to 0$ for $j \to \infty$ and every fixed natural number r. Thus $A^j v \to 0$ as $j \to \infty$. But $Av = \lambda_i v$ and thus $\|A^j v\| = |\lambda_i^j|\, \|v\| \geqq \|v\|$, which is a contradiction. Hence $f_i(A)\, x = 0$.

Therefore $x \in f_n(A)(X)$. Since the spectral radius of the operator A_n is less than 1 the series

$$\sum_{k=0}^{\infty} A^k x = \sum_{k=0}^{\infty} A_n^k x$$

converges.

Note: *Let X be a Banach space. Suppose that U, K are bounded linear operators on X such that K is compact and $\|U\| < 1$. If $x \in X$ then the series $\sum_{n=0}^{\infty} (U + K)^n x$ converges if and only if $(U + K)^n x$ converges weakly to zero as $n \to \infty$.*

Proof. According to Theorem it suffices to prove that if $(U + K)^n x$ converges weakly to zero then it converges to zero. Suppose the contrary. Then there exist $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that $\|(U + K)^{n_k} x\| > \varepsilon$ for each k. Since $(U + K)^n x$ convergences weakly to zero it is bounded according to [T], Theorem 4.4-D. There is a positive constant M such that

$$\|(U + K)^n x\| \leqq M \tag{5}$$

for each natural n. Since $\|U\| < 1$ there exists a natural number n_0 such that

$$\|U\|^{n_0} < \frac{\varepsilon}{4M}\,. \tag{6}$$

According to [DSch], Chapter VI, § 5, Theorem 4 the operator $L = (U + K)^{n_0} - U^{n_0}$ is compact. By virtue of (5) there is a subsequence $\{m_j\}$ of $\{n_k\}$ and $y \in X$ such that $L(U + K)^{m_j - n_0} x$ convergences to y. Since $(L + U^{n_0})(U + K)^{m_j - n_0} x$ converges weakly to zero and $L(U + K)^{m_j - n_0} x$ converges to y, the sequence $U^{n_0}(U + K)^{m_j - n_0} x$ converges weakly to $(-y)$. Now (5), (6) imply

$$\|U^{n_0}(U + K)^{m_j - n_0} x\| < \frac{\varepsilon}{4}\,. \tag{7}$$

If we consider y and $U^{n_0}(U + K)^{m_j - n_0} x$ as elements of the second dual of X we obtain

$\|y\| \leqq \varepsilon/4$. Since $L(U + K)^{m_j - n_0} x$ converges to y there is m_j for which

$$(8) \qquad \|L(U + K)^{m_j - n_0} x\| < \frac{\varepsilon}{2}.$$

From (7), (8) we conclude

$$\|(U + K)^{m_j} x\| < \tfrac{3}{4}\varepsilon,$$

which contradicts $\|(U + K)^{m_j} x\| > \varepsilon$.

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Author's address: Žitná 25, 115 67 Praha 1, Czechoslovakia (Matematický ústav ČSAV).