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Basil K. Papadopoulos

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ON THE SCOTT TOPOLOGY ON THE SET $C(Y, Z)$ OF CONTINUOUS MAPS

BASIL K. PAPADOPOULOS, Xanthi

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It is shown, that if a topology t contains the topology of pointwise convergence and it is splitting on the set $C(Y, Z)$ of continuous maps, then the specialization order of t coincides with the pointwise order induced on $C(Y, Z)$ by the specialization order of Z.

This result is used to prove that when there exists the coarsest jointly continuous topology on $C(Y, Z)$, where Z is an injective T_0 topological space, then this topology is the Scott topology $\sigma(C(Y, Z))$, which is determined by the pointwise order induced on $C(Y, Z)$ by the specialization order of Z.

1. INTRODUCTION

With the following two known propositions the Isbell topology and the bounded-open topology on the set $C(Y, Z)$ are compared with the Scott-topology.

Proposition 1.1. (Proposition 2.10 of [7]). *If Y is a corecompact space and Z is an injective T_0 space, then the Isbell topology T_{is} coincides with the Scott topology $\sigma(C(Y, Z))$, which is determined by the pointwise order induced on $C(Y, Z)$ by the specialization order of Z.*

Proposition 1.2. (Proposition 2.12 of [5]). *If Y is a locally bounded space and Z is an injective T_3-space, then the bounded-open topology T_{eo} coincides with the Scott topology $\sigma(C(Y, Z))$, which is determined by the pointwise order induced on $C(Y, Z)$ by the specialization order of Z.*

In this paper we prove that when there exists the coarsest jointly continuous topology on $C(Y, Z)$, where Z is an injective T_0 space, then this topology is the Scott topology $\sigma(C(Y, Z))$.

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[1]) This paper has been communicated at the Fourth International Conference "Topology and it Applications" (Dubrovnik, Yugoslavia, Sep. 30-Oct. 5, 1985).

Obviously the two propositions mentioned above are corollaries of our result. As a matter of fact, when Y is corecompact the Isbell topology T_{is} is the coarsest jointly continuous topology on $C(Y, Z)$ [5] and when Y is locally bounded and Z regular, the bounded-open topology is the coarsest jointly continuous topology on $C(Y, Z)$ [4].

The worth of the above result relies on the fact that useful property of the Scott topology was found, while a method of comparison of the Scott topology with other topologies is presented. We further note that the considered Scott topology on the set $C(Y, Z)$ is defined by a partial order depending only on the topology of the range space Z. This is attained using Proposition 3.2 in the sequel according to which, if a topology t is splitting on $C(Y, Z)$, where Z is T_0, and contains the topology of pointwise convergence, then the specialization order of t coincides with the pointwise order induced on $C(Y, Z)$ by the specialization order of Z. Notice that all the commonly used topologies on $C(Y, Z)$ satisfy these conditions.

2. PRELIMINARIES

The Scott topology on a complete lattice $(L, \leqq)$ is defined as follows: A subset U of L is *Scott-open* if and only if satisfies the following conditions:

(i) $U = \uparrow U = \{y \in L: x \leqq y \text{ for some } x \in U\}$

(ii) for every directed set $D \subset L$, $\sup D \in U$ implies $D \cap U \neq \emptyset$.

A T_0 space Z is called *injective* if and only if every continuous map $f: X \to Z$ extends continuously to any space Y containing X as a subspace.

The partial order $\leqq$ defined on a T_0 space X by $x \leqq y$ if and only if $x \in \bar{y}$ is called the *specialization order of* X.

If X is an injective T_0 space, then $(X, \leqq)$ is a continuous lattice (with respect to the specialization order) and the Scott topology of this lattice coincides with the topology of X.

The pointwise order $\leqq^*$ induced on $C(Y, Z)$ by the specialization order of Z (with Z a T_0 space) is defined as follows: $f \leqq^* g \Leftrightarrow f(y) \in \overline{g(y)}$, $\forall y \in Y$.

All the above can be found in [2].

A topology t is said to be *splitting on* $C(Y, Z)$, whenever for every space X the continuity of a function $f: X \times Y \to Z$ implies that of its adjoint function $\hat{f}: X \to C_t(Y, Z)$, where $\hat{f}(x)(y) = f(x, y)$ for all x, y i.e. if the exponential injection $E_{XYZ}: C(X \times Y, Z) \to C(X, C_t(Y, Z))$, where $E_{XYZ}(f) = \hat{f}$ is well defined.

A topology t is said to be *jointly continuous* on $C(Y, Z)$, if for every space X the continuity of $\hat{f}: X \to C_t(Y, Z)$ implies that of $f: X \times Y \to Z$ or equivalently if the evaluation function: $e: C_t(Y, Z) \times Y \to Z$, where $e(g, y) = g(y)$, $\forall g \in C(Y, Z)$, $\forall y \in Y$, is continuous.

There exists at most one topology t on the set $C(Y, Z)$ that is both splitting and jointly continuous.

The coarsest jointly continuous topology on $C(Y, Z)$ if it exists, is also (the finest) splitting [6].

Any jointly continuous topology is finer that any splitting one.

The sets of the form $(H, P) = \{f \in C(Y, Z): f^{-1}(P) \in H\}$, where $H \in \Omega(Y)$ ($\Omega(Y)$ is the lattice of open sets of Y with the Day-Kelly-Scott topology [1]) and P is open in Z, generate the Isbell topology T_{is} on $C(Y, Z)$, which is always splitting and contains the compact-open topology T_{co}, [5].

3. MAIN RESULTS

Let us introduce a topology t^* on the set $C(Y, Z)$ as follows: The subbasic neighborhoods of each $f \in C(Y, Z)$ are of the form $\langle f, P\rangle = \{g \in C(Y, Z): f^{-1}(P) \subset \subset g^{-1}(P)\}$, where $P \in O(Z)$ ($O(Z)$ denotes the lattice of open sets of Z).

The proof of the following lemma is obvious and therefore it is ommited.

Lemma 3.1. *The topology t^* is jointly continuous on the set $C(Y, Z)$.*

Proposition 3.2. *If a topology t on the set $C(Y, Z)$, where Z is a T_0 space, is splitting on $C(Y, Z)$ and contains the topology of pointwise convergence T_p, then the specialization order of t coincides with the pointwise order induced on $C(Y, Z)$ by the specialization order of Z.*

Proof. Firstly, we notice that the specialization order of t can be defined. Indeed, from the hypothesis that the topology t contains the topology T_p, $C_t(Y, Z)$ is a T_0 space. Let $f \leqq^* g$ in the pointwise order i.e. for each $y \in Y$, $f(y) \in \overline{g(y)}$. Then, for every $P \in O(Z)$, $f^{-1}(P) \subset g^{-1}(P)$ (1). We choose an arbitrary $T \in t$ as well as an arbitrary $f \in C(Y, Z)$, such that $f \in T$. Since $t \subset t^*$, because t is splitting and t^* is jointly continuous, there exist $P_i \in O(Z)$, such that $f \in \bigcap_{i=1}^{n} \langle f, P_i\rangle =$ $= \bigcap_{i=1}^{n} \{h: f^{-1}(P_i) \subset h^{-1}(P_i)\} \subset T$. By virtue of (1), it follows that, $g \in \bigcap_{i=1}^{n} \{h: f^{-1}(P_i) \subset h^{-1}(P_i)\} \subset T$ thus $g \in T$. Hence every t-open neighborhood T of f contains also g. That means f belongs to t-closure of g (i.e. $f \leqq g$ in the specialization order of t).

Now let $f \in \bar{g}^t$ (i.e. $f \leqq g$ in the specialization order of t). We will prove that for every $y \in Y$, $f(y) \in \overline{g(y)}$ (i.e. $f \leqq^* g$). We take an arbitrary $y \in Y$ as well as an arbitrary $P \in O(Z)$, such that $f(y) \in P$. Since $t \supset T_p$, it follows that $f \in (y, P) \in t$. However $f \in \bar{g}^t$ implies that $g \in (y, P)$ and consequently $g(y) \in P$. Hence $f(y) \in \overline{g(y)}$. This completes the proof.

Corollary 3.3. *Let Y be an arbitrary topological space and Z a T_0 space. Then, the specialization order of the below topologies coincides with the pointwise order induced on $C(Y, Z)$ by the specialization order of Z:*

(i) *topology of pointwise convergence* T_p
(ii) *compact-open topology* T_{co}
(iii) *Isbell topology* T_{is}.
Proof. All these topologies are splitting on $C(Y, Z)$ and $T_p \subset T_{co} \subset T_{is}$.

Theorem 3.4. *If the coarsest jointly continuous topology on the set* $C(Y, Z)$ *exists, where* Z *is an injective* T_0 *topological space, then this is the Scott topology, which is determined by the pointwise order induced on* $C(Y, Z)$ *by the specialization order of* Z.

Proof. We suppose that the coarsest jointly continuous topology t on the set $C(Y, Z)$ exists. This topology will also be the finest splitting and thus the exponential function E_{XYZ}: $C(X \times Y, Z) \to C(X, C_t(Y, Z))$ will be a bijection for every space X. Let X' be a subspace of a space X and let $g: X' \to C_t(Y, Z)$ be a continuous function. By the previous exponential law there is a unique continuous function $\bar{g}: X' \times Y \to Z$ whose adjoint $\hat{\bar{g}}$ is the given function g. Since Z is injective, $\bar{g}$ extends continuously to $G: X \times Y \to Z$. Then, again, the exponential law guarantees the continuity of the adjoint $\hat{G}: X \to C_t(Y, Z)$, which is the required continuous extension of the given function g. So $C_t(Y, Z)$ is an injective space. Because t is jointly continuous we get $t \supset T_p$ (because T_p is splitting) and thus $C_t(Y, Z)$ is T_0 space. Hence, $C_t(Y, Z)$ is continuous lattice in the specialization order of t, which coincides with the Scott topology of this lattice. Finally, by the previous Proposition, we conclude that t coincides with the Scott topology of this lattice, which is determined by the pointwise order induced on $C(Y, Z)$ by the specialization order of Z.

A space Y is corecompact if for every point $y \in Y$ and each open set V containing y there is some open set W bounded in V containing y, [3].

Corollary 3.5. *If the coarsest jointly continuous topology exists on the set* $C(Y, 2)$, *where* 2 *is the Sierpinski space, then* Y *is corecompact.*

Proof. The Sierpinski space 2 is an injective T_0 space [2]. By applying the above Theorem we conclude that this topology is the Scott topology, which coincides with Isbell topology [7], so Y is corecompact [5, Theorem 2.2].

This corollary is known from the fact that the exponential objects in TOP are precisely the corecompact spaces, [8].

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Author's address: Department of Mathematics, Democritus University of Thrace, Xanthi, 67 100, Greece.