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A FUNDAMENTAL ESTIMATE AND AN INEQUALITY OF CACCIOPPOLI'S TYPE FOR NONLINEAR PARABOLIC SYSTEMS OF HIGHER ORDER

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1. INTRODUCTION

Let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \geqq 1$ and boundary sufficiently smooth; let x be a point of $\mathbb{R}^n$, $t \in \mathbb{R}$ and $X = (x, t)$ a point of $\mathbb{R}^n \times \mathbb{R}$. Let N be an integer > 1, $(\ |\ )_k$ and $\|\cdot\|$ the scalar product and the norm in $\mathbb{R}^k$ respectively.

We set $Q = \Omega \times (-T, 0)$, with $T > 0$, and $X_0 = (x^0, t_0)$.

We define

$$B(x^0, \sigma) = \{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\},$$

$$Q(X_0, \sigma) = B(x^0, \sigma) \times (t_0 - \sigma^{2m}, t_0\}$$

and we say that $Q(X_0, \sigma) \Subset Q$ if

$$B(x^0, \sigma) \Subset \Omega \quad \text{and} \quad \sigma^{2m} < t_0 + T \leqq T.$$

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a multi-index and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$; we denote by $\mathcal{R}$, $\mathcal{R}^*$ and $\mathcal{R}'$ respectively the cartesian products $\prod_{|\alpha| \leqq m} \mathbb{R}^N_\alpha$, $\prod_{|\alpha| \leqq m-1} \mathbb{R}^N_\alpha$ and $\prod_{|\alpha| = m} \mathbb{R}^N_\alpha$, while $p = \{p^\alpha\}_{|\alpha| \leqq m}$, $p^* = \{p^\alpha\}_{|\alpha| \leqq m-1}$ and $p' = \{p^\alpha\}_{|\alpha| = m}$, $p^\alpha \in \mathbb{R}^N$, are respectively points of $\mathcal{R}$, $\mathcal{R}^*$ and $\mathcal{R}'$.

If $u: Q \to \mathbb{R}^N$, we set

$$Du = \{D^\alpha u\}_{|\alpha| \leqq m}, \quad \delta u = \{D^\alpha u\}_{|\alpha| \leqq m-1},$$

$$D^{(k)} u = \{D^\alpha u\}_{|\alpha| = k}, \quad k = 1, 2, \dots, m.$$

Let $A^\alpha(X, p)$, $|\alpha| = m$, be vectors of $\mathbb{R}^N$ defined in $Q \times \mathcal{R}$, measurable in X and continuous in p and such that

$$A^\alpha(X, p^*, 0) = 0, \quad |\alpha| = m.$$

Let us consider the nonlinear differential operator

$$E_0 u = (-1)^m \sum_{|\alpha| = m} D^\alpha A^\alpha(X, Du) + \frac{\partial u}{\partial t}. \tag{1.1}$$

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Let us suppose that the vectors $A^\alpha(X, p)$ are differentiable with respect to p' with derivatives $\partial A^\alpha/\partial p_k^\beta$, $|\alpha| = |\beta| = m$, $k = 1, 2, \ldots, N$, measurable in X continuous in p and bounded in $Q \times \mathscr{R}$

$$(1.2) \qquad \left\{\sum_{h,k=1}^{N} \sum_{|\alpha|=m} \sum_{|\beta|=m} \left|\frac{\partial A_h^\alpha}{\partial p_k^\beta}\right|^2\right\}^{1/2} \leqq M\,, \quad \forall (X, p) \in Q \times \mathscr{R}\,.$$

We denote by $A_{\alpha\beta} N \times N$ matrices defined in Q setting

$$A_{\alpha\beta} = \{A_{\alpha\beta}^{hk}\} \quad \text{with} \quad A_{\alpha\beta}^{hk}(X, p) = \int_0^1 \frac{\partial A_h^\alpha(X, p^*, \tau p')}{\partial p_\kappa^\beta}\, d\tau$$

so that, thank to the fact that $A^\alpha(X, p^*, 0) = 0$, we have

$$\sum_{k=1}^{N} \sum_{|\beta|=m} A_{\alpha\beta}^{hk}(X, p)\, p_k^\beta = \int_0^1 \sum_{k=1}^{N} \sum_{|\beta|=m} \frac{\partial A_h^\alpha(X, p^*, \tau p')}{\partial p_k^\beta}\, p_k^\beta\, d\tau =$$

$$= \int_0^1 \frac{\partial A_h^\alpha(X, p^*, \tau p')}{\partial \tau}\, d\tau = A_h^\alpha(X, p)$$

from which it results

$$A^\alpha(X, p) = \sum_{|\beta|=m} A_{\alpha\beta}(X, p)\, p^\beta$$

and

$$E_0 u = (-1)^m \sum_{|\alpha|=m} \sum_{|\beta|=m} D^\alpha[A_{\alpha\beta}(X, Du)\, D^\beta u] + \frac{\partial u}{\partial t}\,.$$

We also suppose that the operator E_0 is strongly parabolic in the following sense: there exists $\nu > 0$ such that

$$(1.4) \qquad \sum_{h,k=1}^{N} \sum_{|\alpha|=m} \sum_{|\beta|=m} \frac{\partial A_h^\alpha}{\partial p_k^\beta}\, \xi_h^\alpha \xi_k^\beta \geqq \nu \sum_{|\alpha|=m} \|\xi^\alpha\|^2$$

for every $(X, p) \in Q \times \mathscr{R}$ and for any system $(\xi^\alpha)_{|\alpha|=m}$ of vectors of $\mathbb{R}^N$. $H^{k,p}(\Omega, \mathbb{R}^N)$ and $H_0^{k,p}(\Omega, \mathbb{R}^N)$, k integer $\geqq 0$, and $p \geqq 1$, are the usual Sobolev space of the vectors $u\colon \Omega \to \mathbb{R}^N$, that, if $p = 2$, we shall simply write with H^k and H_0^k respectively.

If $u \in H^{k,p}(\Omega, \mathbb{R}^N)$, $1 \leqq p < +\infty$, we define

$$|u|_{k,p,\Omega} = \left\{\int_\Omega \Big(\sum_{|\alpha|=k} \|D^\alpha u\|^2\Big)^{p/2} dx\right\}^{1/p},$$

$$\|u\|_{k,p,\Omega} = \left\{\int_\Omega \Big(\sum_{|\alpha|\leqq k} \|D^\alpha u\|^2\Big)^{p/2} dx\right\}^{1/p}$$

that, if $p = 2$, we shall simply write $|\cdot|$ and $\|\cdot\|$ respectively.

The object of the present work is to proof a fundamental inequality [see 2.3] together with an inequality of Caccioppoli's type; these results can be useful to guarantee, as in the case $m = 1$, (see [2]), a fundamental estimate for solutions of the basic systems (see [1]). Therefore the results of this work can constitute a first step in order to obtain, at least for small n, the partial Holder continuity of the solutions to nonlinear parabolic systems of higher order with strictly controlled growth together with an evaluation of Hausdorf measure of singular set; however there are some open problems which we have in mind to examine in a next paper.

The type of solution that we consider is the following:
set

$$a(u, \varphi) = \int_Q \sum_{|\alpha|=m} (A^\alpha(X, Du)/D^\alpha\varphi) - \left(u\Big|\frac{\partial\varphi}{\partial t}\right) \mathrm{d}X,$$

$$W(Q) = L^2(-T, 0, H^m(\Omega, \mathbb{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \mathbb{R}^N)), \tag{1.5}$$

for solution of the system

$$E_0 u = 0 \quad \text{in} \quad Q$$

we mean a vector

$$u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N)) \cap L^\infty(-T, 0, L^2(\Omega, \mathbb{R}^N))$$

so that

$$a(u, \varphi) = 0$$

$$\forall\varphi \in W(Q): \varphi(x, -T) = \varphi(x, 0) = 0\,, \quad \text{in} \quad \Omega$$

or, which is the same, so that

$$\int_Q \left\{\sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(X, Du)\, D^\alpha u \mid D^\beta\varphi) - \left(u\Big|\frac{\partial\varphi}{\partial t}\right)\right\} \mathrm{d}x = 0$$

$$\forall\varphi \in W(Q): \varphi(x, -T) = \varphi(x, 0) = 0\,, \quad \text{in} \quad \Omega\,.$$

2. A FUNDAMENTAL ESTIMATE FOR NONLINEAR PARABOLIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS OF HIGHER ORDER

In this section for the sake of brevity we set $Q(X_0, \sigma) = Q(\sigma)$, $Q(X_0, \sigma)$ being such that: $X_0 \in Q$, $Q(X_0, \sigma) \Subset Q$ and we show that the following theorem holds:

Theorem 2.1. *If $u \in L^2(t_0 - \sigma^2, t_0, H^m(B(\sigma)))$ is solution in $Q(X_0, \sigma)$ of the system*

$$\int_{Q(\sigma)} \left\{\sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(X, Du)\, D^\alpha u \mid D^\beta\varphi) - \left(u\Big|\frac{\partial\varphi}{\partial t}\right)\right\} \mathrm{d}x = 0$$

$$\forall\varphi \in C_0^\infty(Q(\sigma))\,,$$

then it exists $\varepsilon(v, M) \in (0, 1)$ such that $\forall\lambda \in (0, 1)$

$$\int_{Q(\sigma)} \|D^{(m)}u\|^2 \,\mathrm{d}X \leqq c(v, M)\, \lambda^{\varepsilon(n+2m)} \int_{Q(\sigma)} \|D^{(m)}u\| \,\mathrm{d}X\,. \tag{2.1}$$

Fixed $\mu = (M^2 - v^2)/v$, we decompose u as $u = v + w$ where

$w \in L^2(t_0 - \sigma^2, t_0, H_0^m(B(\sigma)))$ is the solution of C.D. problem

$$(2.2) \qquad \int_{Q(\sigma)} \left\{(M + \mu) \sum_{|\alpha|=m} (D^\alpha w \mid D^\alpha \varphi) - \left(w \Big| \frac{\partial \varphi}{\partial t}\right)\right\} dX =$$

$$= \int_{Q(\sigma)} \sum_{|\alpha|=m} \left((M + \mu) D^\alpha u - \sum_{|\beta|=m} A_{\alpha\beta} D^\beta u \mid D^\alpha \varphi\right) dX = 0\,,$$

$$\forall \varphi \in W(Q(\sigma)) \colon \varphi(x, t_0) = 0 \quad \text{in} \quad B(\sigma)$$

while $v \in L^2(t_0 - \sigma^2, t_0, H^m(B(\sigma)))$ is the solution of the system

$$\int_{Q(\sigma)} \left\{(M + \mu) \sum_{|\alpha|=m} (D^\alpha v \mid D^\alpha \varphi) - \left(v \Big| \frac{\partial \varphi}{\partial t}\right)\right\} dX = 0$$

$$\forall \varphi \in C_0^\infty(Q(\sigma))\,.$$

From the linear theory it is known (see Lemma 1.1. p. 68 of [3]) that for any solution $w \in L^2(t_0 - \sigma^2, t_0, H_0^m(B(\sigma)))$ of the system (2.2) we have

$$(2.3) \qquad \int_{Q(\sigma)} \|D^{(m)} w\|^2 \, dX \le K^2(\mu) \int_{Q(\sigma)} \|D^{(m)} u\|^2 \, dX$$

where

$$K(\mu) = \frac{M - \nu + \sqrt{(\mu^2 + M_-^2)}}{M + \mu} \quad \text{with } M_- = \sup \Big\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} \|A_{\alpha\beta}^-\|^2 \Big\}^{1/2}$$

$A_{\alpha\beta}^- = \frac{1}{2}(A_{\alpha\beta} - A_{\alpha\beta}^*)$, $A_{\alpha\beta}^*$ adjoint of $A_{\alpha\beta}$.

Moreover it is known that for the function v the following fundamental estimate holds:

$\forall \lambda \in (0, 1)$

$$(2.4) \qquad \int_{Q(\lambda\sigma)} \sum_{|\alpha|=m} \|D^\alpha v\|^2 \, dX \leqq c(\nu, M)\, \lambda^{n+2m} \int_{Q(\sigma)} \sum_{|\alpha|=m} \|D^\alpha v\|^2 \, dX\,;$$

then from estimates (2.3), (2.4) it follows

$$\begin{aligned}
|D^{(m)} u|_{0,Q(\lambda\sigma)} &\leqq |D^{(m)} w|_{0,Q(\lambda\sigma)} + |D^{(m)} v|_{0,Q(\lambda\sigma)} \leqq \\
&\leqq K(\mu)\, |D^{(m)} u|_{0,Q(\sigma)} + c^{1/2}(\nu, M)\, \lambda^{(n+2m)/2} |D^{(m)} v|_{0,Q(\sigma)} \leqq \\
&\leqq K(\mu)\, |D^{(m)} u|_{0,Q(\sigma)} + c^{1/2}(\nu, M)\, \lambda^{(n+2m)/2} |D^{(m)} u|_{0,Q(\sigma)} + \\
&+ c^{1/2}(\nu, M)\, \lambda^{(n+2m)/2} |D^{(m)} w|^2_{0,Q(\sigma)} \leqq K(\mu)\, |D^{(m)} u|_{0,Q(\sigma)} + \\
&+ c^{1/2}(\nu, M)\, \lambda^{(n+2m)/2} |D^{(m)} u|_{0,Q(\sigma)} + \\
&+ c^{1/2}(\nu, M)\, \lambda^{(n+2m)/2}\, K(\mu)\, |D^{(m)} u|_{0,Q(\sigma)} = \\
&= \{c^{1/2}(\nu, M)\, (1 + K(\mu))\, \lambda^{(n+2m)/2} + K(\mu)\}\, |D^{(m)} u|_{0,Q(\sigma)}\,.
\end{aligned}$$

Since $K(\mu)$, according to the choice we have done for μ, results <1 we may apply the lemma 1.V, p. 12 of [Q] for which it exists $\varepsilon(\nu, M) \in (0, 1)$ such that (2.1) holds; then the proof of Theorem 2.1 is achieved.

In this section we prove the following Caccioppoli's type inequality:

Theorem 3.1. *If* $u \in L^2(-T, 0, H^m(\Omega))$ *is solution in* Q *of the system*

$$\int_Q \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(X, Du)\, D^\alpha u \mid D^\beta \varphi) - \left(u \Big| \frac{\partial \varphi}{\partial t}\right)\Big\} \,dx = 0$$

$\forall \varphi \in C_0^\infty(Q)$

then, $\forall B(x^0, 2\sigma) \Subset \Omega$, $\forall 2a \in (0, T)$, *and for every vector polynomial* $\eta(x)$ *of degree at most* $m - 1$ *it result*:

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|D^{(m)}u\|^2 \,dx \leqq \frac{c(v)}{a} \int_{-2a}^{0} dt \int_{B(2\sigma)} \|u - \eta(x)\|^2 \,dx + \tag{3.2}$$

$$+ \sum_{|\beta|=m} \sum_{\gamma<\beta} \frac{c(v, M)}{\sigma^{2(m-|\gamma|)}} \int_{-2a}^{0} dt \int_{B(2\sigma)} \|D^\gamma(u - \eta(x)\|^2 \,dx\,.$$

Proof. Let $\vartheta(x) \in C_0^\infty(\mathbb{R}^n)$ be a function with the following properties:

$$0 \leqq \vartheta \leqq 1\,, \quad \vartheta = 1 \text{ in } B(\sigma)\,, \quad \vartheta = 0 \text{ in } \mathbb{R}^n \setminus B(\tfrac{3}{2}\sigma) \tag{3.3}$$

$$\|D^\gamma \vartheta\| \leqq c\sigma^{-|\gamma|}\,, \quad \forall \gamma\,.$$

Let $\varrho_p(t)$, with p integer $> 2/a$, $(2a \in (0, T))$, be a function defined on $\mathbb{R}$ as follows

$$\begin{cases} \varrho_p(t) = 1 & \text{if } -a \leqq t \leqq -2/p \\ \varrho_p(t) = 0 & \text{if } t > -1/p \text{ or } t < -2a \\ \varrho_p(t) = t/a + 2 & \text{if } -2a \leqq t \leqq -a \\ \varrho_p(t) = -(p\,.\,t + 1) & \text{if } -2/p \leqq t \leqq -1/p \end{cases} \tag{3.4}$$

Let, at last, $\{g_s(t)\}$ be a sequence of symmetric mollifying functions

$$\begin{cases} g_s(t) \in C(\mathbb{R})\,, \quad g_s(t) \geqq 0\,, \quad g_s(t) = g_s(-t) \\ \operatorname{supp} g_s(t) \subset [-1/s, 1/s] \\ \int_R g_s(t)\,dt = 1\,. \end{cases} \tag{3.5}$$

Since (3.1) is true for any $\varphi \in W(Q)$: $\varphi(x, -T) = \varphi(x, 0) = 0$ in Ω, if $s > \max\{p, 1/(T - 2a)\}$ we can assume in (3.1)

$$\varphi(x) = \vartheta^{2m} \varrho_p[\varrho_p(u - \eta(x)) * g_s]\,. \tag{3.6}$$

Taking into account that

$$D^\beta \varphi(x) = \vartheta^{2m} \varrho_p[\varrho_p D^\beta u * g_s] + \vartheta^m \varrho_p[\sum_{\gamma<\beta} \varrho_p(C_{\beta\gamma}(\vartheta)\, D^\gamma(u - \eta(x)) * g_s]$$

with

$$|C_{\beta\gamma}(\vartheta)| \leqq c\sigma^{-|\gamma|}\,, \quad |\gamma| < m \tag{3.7}$$

we obtain

$$(3.8)\qquad \int_Q (\vartheta^{2m}\varrho_p \sum_{|\alpha|=m}\sum_{|\beta|=m} A_{\alpha\beta}D^\alpha u \mid (D^\beta u\varrho_p * g_s))\, dX \;+$$

$$+ \int_Q (\vartheta^{2m}\varrho_p \sum_{|\alpha|=m}\sum_{|\beta|=m} A_{\alpha\beta}D^\alpha u \Big| \sum_{\gamma<\beta} \varrho_p(C_{\beta\gamma}(\vartheta)\, D^\gamma(u-\eta) * g_s))\, dX \;-$$

$$- \int_Q (u-\eta) \mid \vartheta^{2m}\varrho_p'[(\varrho_p(u-\eta(x)) * g_s)]\, dX =$$

$$= \int_Q (u-\eta) \mid \vartheta^{2m}\varrho_p[(\varrho_p(u-\eta(x)) * g_s]'\, dX .$$

For $g_s(t)$ symmetry the integral at the second member is equal to zero; furthermore when $s \to +\infty$

$$[\varrho(u-\eta(x))] * g \to \varrho_p(u-\eta) \quad \text{in} \quad L^2(-T, 0, H^m(\Omega)).$$

So that, from (3.8), taking the limit for $s \to +\infty$ we obtain

$$(3.9)\qquad \int_Q (\vartheta^{2m}\varrho_p^2 \sum_{|\alpha|=m}\sum_{|\beta|=m} A_{\alpha\beta}D^\alpha u \mid D^\beta u)\, dX =$$

$$= \int_Q (\varrho_p^2\vartheta^m \sum_{|\alpha|=m}\sum_{|\beta|=m} A_{\alpha\beta}D^\alpha u \Big| \sum_{\gamma<\beta} C_{\beta\gamma}(\vartheta)\, D^\gamma(u-\eta(x)) * g_s))\, dX \;-$$

$$- \int_Q \vartheta^{2m}\varrho_p\varrho_p' \|u-\eta(x)\|^2\, dX .$$

Furthermore, by elliptic condition and Hölder inequality from (3.9) we obtain for every $\varepsilon > 0$:

$$\nu \int_Q \vartheta^{2m}\varrho_p^2 \sum_{|\alpha|=m} \|D^\alpha u\|\, dX \leqq m\,.\,\varepsilon \int_Q \varrho_p^2\vartheta^{2m} \sum_{|\alpha|=m} \|D^\alpha u\|^2\, dX \;+$$

$$+\; c(\varepsilon, M) \int_Q \varrho_p^2 \sum_{|\beta|=m}\sum_{\gamma<\beta} \|C_{\beta\gamma}(\vartheta)\|^2\, \|D^\gamma(u-\eta(x))\|^2\, dX \;+$$

$$+ \int_Q \vartheta^2\varrho_p\varrho_p' \|D^\gamma(u-\eta(x))\|^2\, dX.$$

Since from (3.4) it results that

$$\varrho_p\varrho_p' = \begin{cases} \leqq 0 & \text{for } \ t \geqq -2/\varrho \\ = 0 & \text{for } \ t \leqq 2a \ \text{ and for } \ -a \leqq t \leqq -2/\varrho \\ \leqq 1/a & \text{for } \ -2a \leqq t \leqq -a \end{cases}$$

taking into account (3.3), (3.4) and (3.7) and choosing $\varepsilon = \nu/m$ we obtain

$$\int_{-a}^{-2/p} dt \int_{B(\sigma)} \|D^{(m)}u\|^2\, dx \leqq \frac{c(\nu)}{a} \int_{-2a}^{-2/p} dt \int_{B(2\sigma)} \|u-\eta(x)\|^2\, dx \;+$$

$$+ \sum_{|\beta|=m}\sum_{\gamma<\beta} \frac{c(\nu, M)}{\sigma^{2(m-|\gamma|)}} \int_{-2a}^{-2/p} dt \int_{B(2\sigma)} \|D^\gamma(u-\eta(x)\|^2\, dx .$$

Taking the limit for $p \to +\infty$ we have

$$\int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \|D^{(m)}u\|^2 \,\mathrm{d}x \leqq \frac{c(v)}{a} \int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|u - \eta(x)\|^2 \,\mathrm{d}x +$$

$$+ \sum_{|\beta|=m} \sum_{\gamma<\beta} \frac{c(v, M)}{\sigma^{2(m-|\gamma|)}} \int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^{\gamma}(u - \eta(x))\|^2 \,\mathrm{d}x$$

that is the (3.2).

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