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COMPATIBLE TOLERANCES ON GROUPOIDS

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On a groupoid \((G, \cdot)\) a compatible tolerance \(\rho\) is a reflexive and symmetric relation which is a subalgebra of \(G \times G (\rho \leq G \times G)\). If all compatible tolerances of \(G\) are congruences, then \(G\) is called tolerance-trivial, a class \(\mathcal{B}\) of groupoids is tolerance-trivial iff all \(G \in \mathcal{B}\) are tolerance-trivial.

According to the Findlay-Werner's theorem [1], [3], tolerance-trivial and congruence permutable (Mal'cev) varieties coincide but a variety generated by a single tolerance-trivial algebra is not necessarily Mal'cev.

The theory of tolerance-trivial algebras was introduced by I. Chajda and B. Zelinka. In particular they studied tolerance-trivial semigroups. Some main results:

The tolerances-trivial commutative semigroups with at least 3 elements are groups; tolerance-trivial semigroups with at least 3 elements have no bilateral ideal (B. Zelinka [9], [8]; I. Chajda [4]). In his paper [7] B. Pondělíček characterised tolerance-trivial periodic semigroups. Other results containing tolerance-trivial algebras obtained J. Duda, I. Gy. Maurer and other authors [10].

The aim of this article is to study tolerance-trivial groupoids and their classes in order to generalise the above results.

In this article we define and use the notions: left (right, bilateral ideal, proper (left) ideal, maximal, minimal left (right, bilateral) ideal, principal left (right) ideal generated by \(a\), denoted \(\langle a \rangle_L\) in the same way as in the case of the semigroups — and they have analogous properties too. (For example: \(A \subseteq G\) left ideal if for all \(a \in A\) and for all \(g \in G\): \(g \cdot a \in A\), proper left ideal if it is different from \(G\) an \(\emptyset\), — The union and intersection of arbitral system of left ideals are left ideals too.)

For the sake of brevity in the rest of the paper we always write T.-trivial instead of tolerance-trivial.

1. COVERING OF THE GROUPOIDS WITH LEFT (RIGHT) IDEALS

Lemma 1. A covering of a groupoid with a system of proper left (right) ideals \(\{A_i\}_{i \in I}, I \neq \emptyset\) generate a compatible tolerance.

Proof. The required relation is defined as follows: \(a \rho b \iff \exists i \in I, \) such that \(a, b \in A_i(\star)\).
Then \( \varrho \) is clearly reflexive and symmetric. We can write \( \varrho = \bigcup_{i \in I} A_i^2 \) and thus \( \varrho \) is a subgroupoid of \((G, \cdot)\) whence compatible.

**Lemma 2.** Let \( G \) be a \( T \)-trivial groupoid, \( A_1 \) and \( A_2 \) proper left (right) ideals of \( G \) such that \( G = A_1 \cup A_2 \). Then:

(i) \( A_1 \cap A_2 = \emptyset \).

(ii) \( A_1 \) and \( A_2 \) are at the same time maximal and minimal proper left ideals.

(iii) \( G \) has no proper left (right) ideals different from \( A_1 \) and \( A_2 \).

(iv) Each proper left (right) ideal of \( G \) is principal.

(v) Either \( A_1 \) and \( A_2 \) are isomorphic or they are invariant under an arbitrary automorphism \( f \).

**Proof.** (i) Let \( \varrho \) be the relation induced by the covering \( \{A_1, A_2\} \) according to (\(*\)). On applying Lemma 1 we find that \( \varrho \) is a congruence, since \( G \) is \( T \)-trivial. Pick \( a_1 \in A_1 \setminus A_2 \) and \( a_2 \in A_2 \setminus A_1 \). Suppose now that exists \( z \in A_1 \cap A_2 \), then \((a_1, z) \in \varrho\), \((z, a_2) \in \varrho\) but \((a_1, a_2) \notin \varrho\) by the choice of \( a_1 \) and \( a_2 \). This is in contradiction with transitivity of \( \varrho \) and so \( A_1 \cap A_2 = \emptyset \).

(ii) Suppose \( A_1 \) is not maximal, then there exists a proper left ideal \( A \supseteq A_1 \), \( A \neq A_1 \). Since \( \{A, A_2\} \) is also a covering \( A \cap A_2 = \emptyset \), which contradicts the fact \( A \setminus A_1 \subseteq A \cap A_2 \). \( A_2 \) is also maximal by symmetry.

Suppose now that \( A_1 \) is not minimal, i.e. it properly contains a nonvoid left ideal \( B \). But then \( B \cup A_2 \) is proper left ideal containing \( A_2 \) and \( B \neq A_2 \) contradicting the maximality of \( A_1 \). \( A_2 \) (by symmetry) is minimal too.

(iii) Let \( B \) a proper left ideal of \( G \). Clearly \( B \cap A_1 = \emptyset \) or \( B \cap A_2 = \emptyset \). Suppose \( B \cap A_1 = \emptyset \) then by minimality of \( A_1 \) we have \( B \cap A_1 = A_1 \), whence \( A_1 \subseteq B \). Since \( A_1 \) is also maximal \( B = A_1 \). The case \( B \cap A_2 = \emptyset \) is treated similarly.

(iv) Obvious.

(v) Let \( f \) be an automorphism of \( G \). Since \( f(G) = G \) we have that \( f(A_1) \) and \( f(A_2) \) are proper left ideals. Now by (iii) either \( A_1 \) and \( A_2 \) are invarianted or \( f(A_1) = A_2 \) which implies \( A_1 \simeq A_2 \).

**Corollary 1.** If \( G \) is \( T \)-trivial and has a finite covering by proper left ideals, then \( G \) has exactly \( 2 \) proper left ideals. These form a proper covering and Lemma 2 holds.

**Proof.** The assumption implies that \( G \) has also a minimal proper covering, write it \( \{A_1, A_2, \ldots, A_k\}, k \in \mathbb{N} \). Since the ideals of covering are proper \( k \geq 2 \). Let now \( B_1 = A_1 \) and \( B_2 = A_2 \cup A_3 \cup \ldots \cup A_k \). Then \( B_1 \) and \( B_2 \) proper left ideals and follows \( G = B_1 \cup B_2 \). Since \( G \) is \( T \)-trivial, Lemma 2 holds and according to (iii) \( G \) has only \( 2 \) proper left ideals.

**Theorem 1.** If \((G, \cdot)\) is a \( T \)-trivial groupoid and \( \{A_i\}_{i \in I} \), \( I \neq \emptyset \) is a covering of \( G \) with proper left ideals, then the induced relation \( \varrho \) by (\(*\)) is either the total relation on \( G \) or \( I = \{1, 2\} \) and \( \{A_1, A_2\} \) satisfies the conclusion of Lemma 2. Further if \( \varrho \) is the total relation, then for all \( a, b \in G \) there exists \( c \in G \) such that \( a, b \in (c)_L \).
Proof. In Lemma 1 we proved that the relation induced according to \((\ast)\) is a compatible tolerance and since \(G\) is \(T\)-trivial \(\varrho\) is a congruence. Denote the classes of \(\varrho\) by \(\{E_j\}_{j \in J}\).

Obviously \(J \neq \emptyset, E_j \neq \emptyset\) for all \(j \in J\). According to Ju. A. Sreider result [2], page 193, the tolerance blocks of a tolerance generated by a covering can be obtained from members of the given covering, using the operations \(\cap\) and \(\cup\) only. This means that the \(E_j\)-classes are nonvoid left ideals.

Suppose \(|J| = 2\), follows \(E_1 \cup E_2 = G\) and Lemma 2 holds. Now suppose that \(J\) contains more than 2 elements i.e. for all \(j_0 \in J, J \setminus \{j_0\}\) has at least 2 elements. It follows that there exists \(J_1, J_2 \neq J\) such that \(J_1 \cap J_2 = \{j_0\}\) and \(J_1 \cup J_2 = J\). In consequence \(B_1 = \bigcap \{E_j \mid j \in J_1\}\), \(B_2 = \{E_j \mid j \in J_2\}\) are proper left ideals. Moreover \(B_1 \cup B_1 = \{E_j \mid j \in J_1\} = G\), but \(B_1 \cap B_2 \supseteq E_j \neq \emptyset\) which contradicts (i) of Lemma 2. Therefore \(J\) has at most 2 elements. If \(J\) has only one element, then \(E_1 = G\) which means that the congruence \(\varrho\) is the total relation of \(G\).

If \(\{A_j\}_{i \in I}\) generates the total relation, since \(A_i \neq G\), for all \(i \in I, I\) is infinite. Taking in our consideration Corollary 1 it follows that no covering of \(G\) contains 2 left (right) proper ideals. By what has been said it is clear that any proper covering generates the total relation. In particular the covering \(G = \bigcup \{(g)_{L} \mid g \in G\}\) consisting of all principal left ideals generates the total relation, which means that for all \(a, b \in G\) there exists a \(c \in G\) such that \(a, b \in (c)_{L}\).

**Corollary 2.** If \((G)\) is \(T\)-trivial then there exist the following possibilities:

1) \(G\) has two proper left ideals and Lemma 2 holds.
2) All coverings of \(G\) with proper left ideals contain infinite members.
3) There exists a \(g \in G\) such that \((g)_{L} = G\).

Proof. We note that 1) and 2) follows from Theorem 3 and Corollary 1. If both 1) and both 2) is not satisfied on \(G\) it means that \(G\) has not a cover of proper left ideals. In particular \(\bigcup \{(g)_{L} \mid g \in G\}\) is not a proper cover of \(G\), so that there exists a \(g \in G\) with the property \((g)_{L} = G\).

Remark. It seems that in general case 2) in Corollary 2 can be omitted. The problem is that an infinite covering of \(G\) consisting of proper left ideals need not have a minimal subcovering. (e.g. the union of a chain of proper left ideals:

\[ G = \bigcup \{A_i \mid A_i \subseteq G\}, A_i \subseteq A_j \text{ for } i \leq j. \]

If the congruence induced by the covering (as in Theorem 1) has more than one class then there exists a minimal covering since the classes themselves are disjoint ideals.

But a restriction on the structure of \(G\) also can exclude case 2. Probably this happens in the case of semigroups — but the author can neither prove nor disprove it.
Theorem 2. If \((G, \cdot)\) is a T-trivial groupoid with at least 3 elements then either it has no proper bilateral ideals or it has an ideal \(A\) which satisfies the following properties:

(i) \(G \setminus A = \{u\}\) and \(u^2 = u\);

(ii) \(A\) is maximal and minimal;

(iii) \(A\) is the only proper bilateral ideal of \(G\);

(iv) for any \(a \in A\), \(\{u, a\}\) generates \(G\) and \((u)_L = G\), \((u)_R = G\);

(v) \(G\) is direct irreducible (i.e. it can not be presented as a direct product).

Proof. (i) If \(A \subseteq G\) an ideal, the \((G \times A) \cup (A \times G)\) is also a bilateral ideal in \(G \times G\).

Write \(\Delta_G = \{(g, g) \mid g \in G\}\) and \(R = \Delta_G \cup (G \times A) \cup (A \times G)\). Now \(R\) is a subgroupoid of \(G \times G\) and by definition it is reflexive and symmetric, so \(R\) is a compatible tolerance. Since \((G, \cdot)\) is T-trivial, \(R\) is a congruence.

Suppose \(x, y \in G \setminus A\), \(x \neq y\), then for all \(a \in A\): \((x, a) \in R, (a, y) \in R\) but \((x, y) \notin R\) which contradicts the fact that \(R\) is a congruence therefore \(G \setminus A\) has a single element which we denote by \(u\). In [5] I. Chajda proved that in a T-trivial algebra \((A, F)\) for all \(a \in A\) there exists \(f \in F\) such that \(a = f(a_1, \ldots, a_n)\) for some \(a_1, \ldots, a_n \in A\).

In our case this means that there exists \(u_1, u_2 \in G\) such that \(u_1 \cdot u_2 = u\). But \(u_1, u_2 \notin A\) implies \(u_1 = u_2 = u\), thus \(u^2 = u\).

(ii) The fact that \(A\) is maximal is obvious. To see that it is minimal, let \(A' \subseteq A\) be an ideal of \(G\). Then \(G \setminus A'\) has a single element (namely the same \(u\)) by (i). Thus \(A' = A\) and so \(A\) is minimal.

(iv) If \(G_0\) is a subgroupoid of \(G\) such that \(G_0 \cup A = G\), then \(\pi = (G_0 \times G_0) \cup (A \times A)\) is a congruence on \(G\).

Indeed, by definition \(\pi\) is reflexive and symmetric and \(\pi = (G_0 \times G_0) \cup (A \times A)\) is a subgroupoid of \(G \times G\). Since \(G\) is T-trivial \(\pi\) is a congruence. If there exists \(z \in G_0 \cap A\), then for all \(u \in G\) and \(a \in A \setminus G_0\), we have \((u, z) \in \pi\), \((z, a) \in \pi\) but \((u, a) \notin \pi\).

There are two possibilities to evade the contradiction; either \(G_0 \cap A = \emptyset\) or \(A \subseteq G_0\). In the first case \(G_0 = \{u\}\) and in the latter \(G_0 = G\). Now let \(a \in A\), and put \(G_0 = \langle a, u \rangle\). We find that \(\langle a, u \rangle = G\). Consider now \(G_0 = (u)_L\). Since \((u)_L \cap A = \emptyset\) (for all \(a \in A\), \(a \cdot u \in (u)_L \cap A\)) we have: \((u)_L = G\). Symmetrically we obtain \((u)_R = G\).

(iii) Let \(A'\) be another ideal of \(G\). Since \(A' \not\subseteq A\), we have \(u \in A'\) by (i). We obtain \(A' \supseteq (u)_L = G\).

(v) For an arbitrary congruence \(\theta\) and for the idempotent \(u \in G\), \(\theta[u]\) is a subgroupoid (see I. Chajda [4]). By the proof of (iv) we find that either \(\theta[u] = \{u\}\) or \(\theta[u] = G\). In the first case \(\theta \subseteq \{u, u\} \cup (A \times A)\), (where \(\{u\} = G \setminus A\), while in the latter \(\theta = G \times G\). It is well-known that \(G\) can be presented as a product of two (non-trivial) groupoids \(G_1\) and \(G_2\) if there exist the non-total congruences \(\theta_1\)
and \( \theta_2 \) such that \( \theta_1 \lor \theta_2 = G \times G \) and \( \theta_1 \land \theta_2 = \Delta_G \). The fact that \( \theta_1 \) and \( \theta_2 \) are non-total implies \( \theta_i \subseteq \{(u, u)\} \cup (A \times A), \ i \in \{1, 2\} \); but then \( \theta_1 \lor \theta_2 \subseteq \{(u, u)\} \cup (A \times A) \) — a contradiction.

Next we list a few consequences. The first we have already verified in the course of proving (iv):

**Corollary 1.** If \((G, \cdot)\) is a T.-trivial groupoid containing a proper bilateral ideal \(A\), and a subgroupoid \(G_0\); then \(G_0 \cup A = G\) implies either \(G_0 \cap A = \emptyset\) or \(G_0 = G\).

**Corollary 2.** If \((G, \cdot)\) cannot be generated by 2 elements and \((G, \cdot)\) is T.-trivial, then \(G\) has no proper bilateral ideals.

**Corollary 3.** If \((G, \cdot)\) is a T.-trivial groupoid and further \(G\) has a neutral element, then \(G\) either has a single proper ideal which is a maximal and minimal subgroupoid of \(G\) at the same time or \((G, \cdot)\) has no proper ideal at all.

**Proof.** Let \(e\) be the neutral element of the T.-trivial groupoid \(G\), and \(A\) a proper bilateral ideal of \(G\), obviously \(e \notin A\) (otherwise \(A = G\)). According to (i) of Theorem 1, we have \(G \setminus A = \{e\}\) so \(A\) is a maximal subgroupoid.

Let \(g \in A\), then \(\langle g \rangle \subseteq A\). Since \(e \cdot \langle g \rangle = \langle g \rangle \cdot e = \langle g \rangle\), \(G_0 = \{e\} \cup \langle g \rangle\) is a subgroupoid and \(G_0 \cup A = G\), while \(G_0 \cap A = \langle g \rangle \neq \emptyset\). But then by Corollary 1 of Theorem 2 it follows that \(G_0 = G\), and so \(\langle g \rangle = A\). Since \(g\) is an arbitrary element of \(A\), we find that \(A\) is a minimal subgroupoid of \(G\).

**Corollary 4.** If both \((G, \cdot)\) and \((G \times G, \cdot)\) are T.-trivial \(G\) has no proper bilateral ideal.

**Proof.** If \(G\) has a single element, the claim is obvious, if \(G\) has at least two elements \(G \times G\) has at least 4 and so (i) and (v) of Theorem 2 can be applied.

**Theorem 3.** Let \((G, \cdot)\) be a T.-trivial groupoid with at least 3 elements and assume that \(G\) is covered by subsemigroups, each of cardinality \(> 1\), then \(G\) has no proper bilateral ideal.

**Proof.** Let’s suppose that \(G\) has a proper bilateral ideal \(A\). According to (i) of theorem 2, \(G \setminus A\) has a single element say \(u\). Denote by \(S_u\) the subsemigroup of \(G\) which contains \(u\). Since \(S_u\) has at least 2 elements \(S_u \cap A = \emptyset\) and so \(S_u = G\) by Corollary 1 to Theorem 2. Thus \((G, \cdot)\) is a semigroup. Since \(u^2 = u\) and „·“ is associative \((u)_L = G \cdot u\) and \((u)_R = u \cdot G\). But according to (iv) of Theorem 2, \(G = G \cdot u = u \cdot G, i.e.\) for all \(x \in G\) there exist \(k, l \in G\) such that \(x = k \cdot u\) and \(x = u \cdot l\). But in this case \(x \cdot u = u \cdot x = x\) whence \(u\) is a neutral element of \(G\). By applying Corollary 3 to Th. 2 we find that \(\langle a \rangle = A\) in consequence \(A\) has no subgroupoid which is proper left or right ideal of \(A\). It means that \((A, \cdot)\) is a group. Denote by \(e\) the neutral element of this group. Then \(\{e\} = \langle e \rangle = A\) and so \(G = \{u, e\}\) which contradicts the assumption that \(G\) has at least 3 elements.

**Corollary 5.** If \((G, \cdot)\) is a T.-trivial semigroup with at least 3 elements, \(G\) has no proper bilateral ideal.
**Proposition.** If \((G, \cdot)\) is a \(T\)-trivial groupoid with at least 3 elements and \(G\) has a congruence with only 2 classes which are also subgroupoids, then there are 2 possibilities:

**Case 1.** These classes are at the same time maximal and minimal unilateral ideals which satisfy the conclusions of Lemma 2.

**Case 2.** One of them has only one element and the other is a bilateral ideal which satisfies the conclusions of Theorem 2.

**Proof.** Let \(\varepsilon\) be the congruence with classes \(E_1\) and \(E_2\). Let \(a_1 \in E_1\) and \(a_2 \in E_2\). Then either \(a_1 \cdot a_2 \in E_1\) or \(a_1 \cdot a_2 \in E_2\). Suppose \(a_1 \cdot a_2 \in E_1\). Since \(E_2\) is a congruence class, according to Malcev's result ([6] page 3) for all algebraic functions \(t\) on \(G\) either \(t(E_2) \subseteq E_2\) or \(t(E_2) \cap E_2 = \emptyset\) (in our case the second relation means \(t(E_2) \subseteq E_1\)). In particular it holds for the \(t_0(x) = a_1 \cdot x\). Since \(t_0(a_2) \in E_2\) we have \(t_0(E_2) \subseteq E_2\) so that \(a_1 \cdot x \in E_2\) for all \(x \in E_2\). Consider now the algebraic function \(t_1(y) = y \cdot x\) and the class \(E_1\); since \(a_1 \cdot x \in E_2\) for an arbitrary but fixed \(x \in E_2\) it follows that \(t_1(y) \in E_2\) for any \(y \in E_1\). Thus \(y \cdot x \in E_2\) for all \(y \in E_1\) and \(x \in E_2\). Moreover, since \((E_2, \cdot)\) is subgroupoid \(y \cdot x \in E_2\) for all \(y \in G\) and \(x \in E_2\), i.e. \(E_2\) is left ideal.

If \(a_2 \cdot a_1\) belongs to \(E_1\) on changing the roles of \(a_1\) and \(a_2\) we can show that \(E_1\) is a left ideal. Then the system \(\{E_1, E_2\}\) satisfies the hypotheses of Lemma 2 and so the case 1 occurs.

If \(a_2 \cdot a_1\) is also in \(E_2\) then by a symmetrical argument as before we can show \(E_2\) is a right ideal. Thus \(E_2\) is bilateral and case 2 occurs.

**Corollary 6.** Let \((G, \cdot)\) be a \(T\)-trivial idempotent groupoid with at least 3 elements which satisfies:

(i) \(G\) has no proper bilateral ideal.

(ii) The number of maximal or equivalently minimal one-sided ideals of \(G\) is not equal to 2.

Then all congruences different from \(G \times G\) on \(G\) have at least 3 classes.

**Proof.** If \(\varrho\) is a congruence on \((G, \cdot)\), since \((G, \cdot)\) is idempotent all classes of \(\varrho\) are subgroupoids. Now the claim follows by the proposition.

### 3. COMPATIBLE TOLERANCE ON CLASSES OF GROUPOIDS

In what follows by \(C(\varrho)\) we mean the system of subgroupoids of the class \(\varrho\) of groupoids.

**Theorem 4.** Suppose that the class \(\varrho\) of groupoids satisfies:

(i) \(\varrho\) is \(T\)-trivial.

(ii) For every \(G \in \varrho\), any subdirect square of \(G\) is \(T\)-trivial.

Then no \(G \in \varrho\) contains proper left or proper right ideals.
Proof. Let be $G \in \mathcal{G}$ and $A$ a proper left ideal of $G$. It is easy to see that the left ideal $B = (G \times A) \cup (A \times G)$ of $G \times G$ is subdirect square of $G$. Thus $B$ is a T-trivial groupoid. But $G \times A$ and $A \times G$ are proper left ideals of $B$ and $(G \times A) \cap (A \times G) = A \times A \neq \emptyset$, which contradicts (i) of Lemma 2.

Corollary 1. If $\mathcal{G}$ is a Mal'cev variety then no $G \in \mathcal{G}$ contains proper left or right ideals.

Theorem 5. Let $\mathcal{G}$ be a class of groupoids which satisfies:

(i) $\mathcal{G}$ is T-trivial.

(ii) $C(\mathcal{G}) \subseteq \mathcal{G}$.

Then for every $G \in \mathcal{G}(G \neq \emptyset)$ precisely one of the following conditions holds:

1) The left ideals of $G$ form a chain.

2) $G$ has exactly 2 disjoint left ideals $A_1$ and $A_2$; apart from them the left ideals of $G$ form a chain and moreover, for each left ideal $A \neq \emptyset$ of $G$ different from $A_1$ and $A_2$ we have $A_1 \cup A_2 \subseteq A$. The lattice-structure of the left (right) ideals of $G$ is given by:

![Diagram](image)

3) $G$ has no proper left ideal.

Proof. Let $A_1$ and $A_2$ be two nonvoid left ideals of $G \notin \mathcal{G}$. If neither $A_1 \subseteq A_2$ nor $A_2 \subseteq A_1$, then for the T-trivial groupoid $B = A_1 \cup A_2$, $A_1$ and $A_2$ form a covering. But according to (i) of Lemma 2: $A_1 \cap A_2 = \emptyset$ and according to (ii) of Lemma 2. $A_1$ and $A_2$ are minimal in $G$. (If $A \lhd G$ and $A \subseteq B$, then $A$ is also a left ideal of $B$.) In conclusion if $G$ has no 2 minimal left ideals, case 1 or case 3 occurs.

$G$ cannot contain more then 2 disjoint left ideals: Let $A_1$, $A_2$ and $A$ be pairwise disjoint left ideals, then $B' = A_1 \cup A_2 \cup A$ is a T-trivial subgroupoid with a covering of 3 proper left ideals in contradiction with Corollary 2 of Lemma 2.

Assume now, that $G$ contains 2 minimal left ideals, namely $A_1$ and $A_2$. For a given left ideal $A$ of $G(A \neq A_1$, $A \neq A_2$) we have either $A_1 \subseteq A$ or $A_2 \subseteq A$ or both. Suppose $A_1 \subseteq A$ but $A_2 \nsubseteq A$, then $A_2$ and $A$ are disjoint since $\emptyset \neq A \cap A_2 = A_2$ contradicts $A_2$ is minimal. So applying Lemma 2 to the T-trivial subgroupoid $B_1 = A \cup A_2$ follows that $A_1$ is minimal. Thus $A_1 = A$ contradicting the choice of $A$. The case $A_2 \subseteq A$ but $A_1 \nsubseteq A$ is treated similarly.
We have verified that $A_1 \cup A_2 \subseteq A$. Now let $A$ and $B$ be 2 proper left ideals different from $A_1, A_2$ and each other. Then $A \cap B \supseteq A_1 \cup A_2 \neq \emptyset$, So $D = A \cup B$ is also a T.-trivial groupoid. Repeating the above arguments we get either $A \subseteq B$ or $B \subseteq A$. In conclusion case 2 occurs.

**Observation.** Let $\mathcal{G}$ satisfy the conditions of the Theorem 5 and assume that $G \in \mathcal{G}$ has only a finite number of left (right) ideals. Further, let $A_1$ and $A_2$ be as in the statement of the theorem (i.e. there are minimal left ideals). Then the following holds:

(i) Each proper left ideal of $G$ is a principal left ideal except $A_1 \cup A_2$.
(ii) Every left ideal of $G$ is invariant under any automorphism of $G$ except possibly $A_1$ and $A_2$.

**Proof.** (i) Since $A_1$ and $A_2$ are minimal left ideals it is well-known that they are principal left ideals. Let $B$ be a proper left ideal of $G$ different from $A_1 \cup A_2$, $A_1$ and $A_2$. Then $A_1 \cup A_2 \subseteq B$. Since $G$ has a finite number of left ideals, there exists a maximal left ideal of $G$ with the property $A \nsubseteq B$. Pick $a \in B \setminus A$. Then $(a)_L \subseteq B$, but $(a)_L \nsubseteq A$. From Theorem 5 it follows that $A \subseteq (a)_L$ whence $(a)_L = B$ and thus $B$ is principal.

(ii) If $G$ contains two disjoint minimal left ideals $A_1$ and $A_2$ we have that $f(A_1)$ and $f(A_2)$ are also minimal and disjoint, for all $f \in \text{Aut } G$. Thus either $f(A_1) = A_1$ and $f(A_2) = A_2$ or $f(A_1) = A_2$ and $f(A_2) = A_1$. In both cases: $f(A_1 \cup A_2) = A_1 \cup A_2$.

Now let $A$ be a left ideal of $G$ different from $\emptyset, A_1$ and $A_2$. We want to show $f(A) = A$. According to Theorem 5 we have $f(A) \subseteq A$ or $A \subseteq f(A)$. Assuming the latter we find $A = f(A)$ since $f$ induces a strictly order-preserving map of the finite chain of left ideals containing $A$ into itself (i.e. the identity map). If we assume $f(A) \subseteq A$ the above argument applies to $f^{-1}$.

If the ideals of $G$ form a finite chain, any automorphism $f$ induces also an order-preserving bijection of the chain onto itself (i.e. the identity map) and the claim follows.

4. STRONGLY T.-TRIVIAL GROUPOIDS

**Definition.** We call a groupoid $(G, \cdot)$ strongly T.-trivial iff all subgroupoids of $G$ including $(G, \cdot)$ itself are T.-trivial.

**Observation 1.** If $(G, \cdot)$ is a finite strongly T.-trivial groupoid, then $G$ satisfies the conclusions of Theorem 3 and Observation (§ 3).

**Observation 2.** If $G$ is strongly T.-trivial then any direct decomposition of $G$ as a direct product of groupoids contains at most one groupoid which has a proper left (right) ideal.

**Proof.** Let $G = A_1 \times A_2 \times \ldots \times A_n$. If $B_1$ and $B_2$ are proper left ideals of two different factors, without loss of generality we may assume that $B_1 \subseteq A_1$ and
$B_2 \leq_L A_2.$ But then $B_1 \times A_2 \times \ldots \times A_n$, and $A_1 \times B_2 \times \ldots \times A_n$ are also left ideals with non-empty intersection, and certainly neither of them contains the other—contradicting theorem 5.

**Theorem 6.** If a strongly $T$-trivial groupoid $G$ with at least 3 elements contains a neutral element $e$ then:

(i) For all $g \in G$ with the property $e \notin \langle g \rangle$, $\langle g \rangle$ is a minimal subgroupoid of $G$.

(ii) For all $x, y \in G$ which do not belong to the same minimal subgroupoid: $e \in \langle x, y \rangle$.

(iii) $G$ contains at most one proper left ideal (right ideal, bilateral ideal) and that is a minimal subgroupoid in $G$.

(iv) If $G$ contains both proper left ideal and proper right ideal, then they are equal and form a single bilateral ideal of $G$, which satisfies Corollary 3 of Theorem 2.

**Proof.** (i) Let $G_0 \neq 0$ a subgroupoid of $G$, then $\{ e \} \cup G_0$ is also a subgroupoid of $G$ and $G_0$ is bilateral ideal in $\{ e \} \cup G_0$. So 2 cases are possible: $e \in G_0$ or $e \notin G_0$ — in the last case Corollary 3 of Theorem 2 implies that $G_0$ is a minimal subgroupoid. If $G_0 = \langle g \rangle$ for a $g \in G$, $e \notin \langle g \rangle$ we get the statement (i).

(ii) If $G_0 = \langle x, y \rangle$, and $\langle x \rangle \neq \langle y \rangle$ then $G_0$ cannot be minimal subgroupoid of $G$ so that $e \in \langle x, y \rangle$.

(iii) If $B$ is a left (right, bilateral) proper ideal of $G$ then $e \notin B$ (otherwise $B = G$). If $B_0$ denote the union of all proper left ideals of $G$, then $e \notin B_0$. Since $e \cdot B_0 = B_0 \cdot e = B_0$, $B_0$ is a proper bilateral ideal in $\{ e \} \cup B_0$. If $B_0$ has at least 2 elements, $B_0$ is minimal subgroupoid. (If $B_0$ has only one element it is obvious). Since $B_0$ is also a minimal left ideal (right, bilateral ideal), $G$ has no more than a single left ideal (right ideal, bilateral ideal).

(iv) If $G$ contains a proper left ideal $B$ and a proper right ideal $J$ then they are minimal subgroupoids. Since for all $b \in B$ and for all $j \in J$ we have $b \cdot j \in B \cap J$, $B \cap J$ is also a nonvoid subgroupoid, which follows, $B = B \cap J = J$. So $B$ and $J$ form a bilateral ideal, and the hypotheses of Theorem 2 are satisfied.

**Corollary.** If $(G, \cdot)$ is a groupoid with a neutral element which belongs to a Mal'cev variety, for all the $g \in G$ we get: $e \in \langle g \rangle$.

**Proof.** Put $G_0 = \langle g \rangle$ in the proof of (i) of Theorem 6. Supposing that $e \notin \langle g \rangle$, we get: $\langle g \rangle$ is a bilateral ideal in the subgroupoid $\{ e \} \cup \langle g \rangle$. Since it belongs also to a Mal'cev variety, this case is impossible according to Theorem 4.

**Bibliography**


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