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ON THE EXISTENCE OF GENERALIZED SOLUTIONS
OF NONLINEAR FIRST ORDER PARTIAL
DIFFERENTIAL-FUNCTIONAL EQUATIONS
IN TWO INDEPENDENT VARIABLES

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1. INTRODUCTION

Let \( a_0 > 0, B = [-b_0, 0] \times [-b_1, b_1], \) where \( b_0, b_1 \in R_+, R_+ = [0, +\infty) \). For any function \( z: [-b_0, a_0] \times R \to R \) and for a fixed \( (x, y) \in [0, a_0] \times R \) we define the function \( z_{(x,y)}: B \to R \) by \( z_{(x,y)}(t, s) = z(x + t, y + s), \) \( (t, s) \in B \). For any metric spaces \( X, Y \) by \( C(X, Y) \) we denote the set of all continuous functions defined on \( X \) and taking values in \( Y \).

Suppose that \( f: [0, a_0] \times R \times R \times C(B, R) \times R \to R, \phi: [-b_0, 0] \times R \to R \) and let us consider the following Cauchy problem for the nonlinear differential-functional equation of the first order

(1) \[ D_z z(x, y) = f(x, y, z(x, y), z_{(x,y)}, D_z z(x, y)), \]

(2) \[ z(x, y) = \varphi(x, y), \quad (x, y) \in [-b_0, 0] \times R. \]

A function \( u \in C([-b_0, a_0] \times R, R) \) is a generalized solution of (1), (2) if

(i) \( u \) satisfies the Lipschitz condition on \([0, a_0] \times R,\)

(ii) there exists a function \( \lambda \in C([0, a_0], R_+) \) such that \( l^{-2}[u(x, y + l) - u(x, y - l)] \leq \lambda(x), \) for \((x, y) \in (0, a_0) \times R, l \in R, l \neq 0,\)

(iii) \( u \) satisfies (1) a.e. ("almost everywhere") on \([0, a_0] \times R\) and the initial condition (2) for all \((x, y) \in [-b_0, 0] \times R.\)

Remark 1. If we omit the condition (ii) in the above definition, then the solution is not unique. An adequate example for \( f \) without a functional argument is given in [16].

Generalized solutions of nonlinear first order partial differential equations have been investigated in a large number of papers by various authors. Theorems of existence, uniqueness and continuous dependence upon Cauchy or boundary data for quasilinear systems have been given by L. Cesari [7], [8], P. Bassanini [1]—[3] and P. Pucci [18]. Quasilinear differential-integral systems and systems with a retarded argument are considered in [4], [13], [14]. Nonlinear differential equations
have been studied by M. Cinquini-Cibrario, S. Cinquini [9]. Generalized solutions of quasilinear and nonlinear equations with operators of the Volterra type are investigated in [20]–[22]. Additional bibliographical information may be found in [19].

Generalized solutions of nonlinear equations are also investigated in the case when assumptions for given functions are sufficient for existence of classical solutions (of class $C^1$). For classical solutions we can prove only a local existence and therefore to obtain theorems of a global existence we need generalized solutions. Theorems of this type have been given by S. N. Kruzhkov [16] and for equations with a retarded argument by Z. Kamont, S. Zacharek [15].

Classical solutions of nonlinear differential-functional equations or equations with a retarded argument are discussed in [5], [6], [10]–[12].

In this paper we prove the global existence of generalized solutions of (1), (2) extending the results of paper [16]. The proof is based on the difference method (see also [17]).

2. ASSUMPTIONS AND DEFINITIONS

We denote by $C_{0+L}(B, R)$ a set of all continuous functions from $B$ to $R$ which satisfy the Lipschitz condition on $B$. Furthermore, for any $t \in R_+$ let $C_{0+L}(B, R, t) = \{ u \in C_{0+L}(B, R) : \| u \|_{0+L} = \| u \|_0 + \| u \|_L \leq t \}$, where

$$\| u \|_0 = \sup \{ |u(s, v)| : (s, v) \in B \},$$

$$\| u \|_L = \sup \{ [ |s - \bar{s}| + |v - \bar{v}| ]^{-1} |u(s, v) - u(\bar{s}, \bar{v})| : (s, v), (\bar{s}, \bar{v}) \in B \}.$$

Suppose that the function $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$ of the variables $(x, y, p, w, q)$ is of class $C^2$. By $D_x f, D_y f, D_{p} f, D_{p}^2 f, D_y^2 f, D_{w} f, D_{w}^2 f, D_{w}^2 f$ we denote first or second order partial derivatives of $f$. $D_{w} f$ is the Frechet derivative of $f$ i.e. $D_{w} f(x, y, p, w, q) \in \mathcal{L}(C(B, R), R)$, where $\mathcal{L}(X, Y)$ denotes a set of all linear operators from $X$ to $Y$. Symbols $D_{w}^2 f, D_{w}^2 f, D_{w}^2 f$ have the same meaning as $D_{w} f$ while $D_{w}^2 f$ denotes the second order Frechet derivative i.e. $D_{w}^2 f(x, y, p, w, q) \in \mathcal{L}(C(B, R), \mathcal{L}(C(B, R), R))$.

Assumption H. Suppose that

1° $\varphi \in C([-b_0, 0] \times R, R)$ and there are constants $\tilde{M}, \tilde{L} \in R_+$ such that for all $(x, y), (\tilde{x}, \tilde{y}) \in [-b_0, 0] \times R$ we have

$$|\varphi(x, y)| \leq \tilde{M}, \quad |\varphi(x, y) - \varphi(\tilde{x}, \tilde{y})| \leq \tilde{L}|y - \tilde{y}|;$$

2° if $b_0 > 0$, then there is a constant $\tilde{K} \in R_+$ such that for all $(x, y) \in [-b_0, 0] \times R$, $l \in R$, $l \neq 0$ we have $l^{-2}[\varphi(\varphi(x, y + l) - \varphi(x, y)) - \varphi(x, y - l)] \leq \tilde{K};$

3° $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$ is of class $C^2$;

4° there are a constant $N \geq \tilde{M}$ and a nondecreasing function $V \in C([\tilde{M}, N], R_+)$, $\int_0^a dt/V(t) \geq a_0$, such that for all $t \in R_+$, $(x, y, p, w, q) \in [0, a_0] \times R \times [-t, t] \times C(B, R, t) \times R$ we have $|f(x, y, p, w, q)| \leq V(t);$
there are constants $N_1 \geq L$, $A > 0$ and a nondecreasing function $W \in C([\mathcal{L}, N_1], R_+)$, $Q_t^2 \, dt \{ [(2t + 1) W(N + 3t)] \geq a_0$ such that for all $t \in R_+$, $(x, y, p, w, q) \in [0, a_0] \times R \times [-N, N] \times C_{0+L}(B, R, t) \times R$ we have

\[
|D_q f(x, y, p, w, q)| \leq A, \quad |D_y f(x, y, p, w, q)| \leq W(t),
\]

\[
|D_p f(x, y, p, w, q)| \leq W(t), \quad \|D_w f(x, y, p, w, q)\| \leq W(t);
\]

6° for all $(x, y, p, w, q) \in [0, a_0] \times R \times [-N, N] \times C_{0+L}(B, R, R + N + 3N_1) \times \times [-N, N_1], \bar{w} \in C(B, R), \bar{w} \geq 0, we have $D_\bar{w} f(x, y, p, w, q)(\bar{w}) \geq 0$;

7° the derivatives $D_{x}^2 f, D_{y}^2 f, D_{w}^2 f, D_{p}^2 f, D_{w}^2 f, D_{p}^2 f, D_{w}^2 f, D_{w}^2 f$ are bounded and $D_{ww}^2 f \leq 0$ on $[0, a_0] \times R \times [-N, N] \times C_{0+L}(B, R, R + N + 3N_1) \times \times [-N, N_1];$

8° if $b_0 > 0$, then there are constants $\delta \in (0, a_0], \mu > 0 such that $D_{ww}^2 f \leq -\mu$ on $[0, \delta] \times R \times [-N, N] \times C_{0+L}(B, R, R + N + 3N_1) \times \times [-N, N_1].$

Let $Z$ be a set of all integers, and let $2Z$ be a set of all even numbers. Now we introduce a difference scheme for (1), (2). For $h, k > 0$ we define $x^{(i)} = ih, i = 0, 1, \ldots, n_0$, $n_0 h = a_0$ and $y^{(j)} = jk, j \in Z$. If $b_0 > 0$, then there is an integer $n_1 > 0$ such that $-n_1 h \leq b_0 < (n_1 + 1) h$. We define $x^{(i)} = ih, i = -n_1 + 1, \ldots, -1$, and $x^{(-n_1)} = -b_0$.

Let $U = \{(h, k): A < k|h|\}, E^* = \{(x^{(i)}, y^{(j)}): i = 0, \ldots, n_0, j \in Z\}$. For $i = 0, \ldots, n_0 - 1, j \in 2Z$ we write $P_{ij} = [x^{(i)}, x^{(i+1)}] \times [y^{(j-1)}, y^{(j+1)}], Q_i = \bigcup_{j \in 2Z} P_{ij}$. If $v: E^* \to R$, then we denote $v^{(i,j)} = v(x^{(i)}, y^{(j)}), i = 0, \ldots, n_0, j \in Z$.

Let $\Delta_0, \Delta_1$ be operators defined by

\[
\Delta_0 v^{(i,j)} = \frac{1}{h} [v^{(i+1,j)} - v^{(i,j)}], \quad \Delta_1 v^{(i,j)} = \frac{1}{2k} [v^{(i,j+1)} - v^{(i,j-1)}].
\]

Furthermore, let $\Delta_{ij} v^{(i,j)} = \Delta_i \Delta_j v^{(i,j)}, i', j' = 0, 1$.

Let $\phi_{hk} : [-b_0, 0] \times R \to R$ be a function defined in the following way:

(i) If $b_0 > 0$, then for each $(x, y) \in [-b_0, 0] \times R$ there are $i, -n_1 \leq i < 0$ and $j \in 2Z$ such that $(x, y) \in [x^{(i)}, x^{(i+1)}] \times [y^{(j-1)}, y^{(j+1)}]$. Then we write $\phi_{hk}(x, y) = \phi(x^{(i)}, y^{(j-1)}) + (x - x^{(i)}) \Delta_0 \phi(x^{(i)}, y^{(j-1)}) + (y - y^{(j-1)}) \Delta_1 \phi(x^{(i)}, y^{(j-1)}) +$ $+(x - x^{(i)})(y - y^{(j-1)}) \Delta_{01} \phi(x^{(i)}, y^{(j-1)}).

(ii) If $b_0 = 0$, then for each $y \in R$ there is $j \in 2Z$ such that $y \in [y^{(j-1)}, y^{(j+1)}]$.

Then we write $\phi_{hk}(y) = \phi(y^{(j-1)}) + (y - y^{(j-1)}) \Delta_1 \phi(y^{(j-1)}).

For any $(h, k) \in U$ let us define the function $u_{hk} : [-b_0, 0] \times R \to R$. We use the mathematical induction in the following way:

(i) Let $v^{(0,j)} = \phi(0, y^{(j)}), j \in Z$ and $u_{hk}(x, y) = \phi_{hk}(x, y)$ for $(x, y) \in [-b_0, 0] \times R$.

(ii) If for some $i, 0 \leq i \leq n_0 - 1$ we have defined $v^{(i,j)}$, $j \in Z$ and $u_{hk}$ on $([-b_0, 0] \times R) \cup Q_0 \cup \ldots \cup Q_{i-1}$, then

\[
v^{(i+1,j)} = \frac{1}{2} [v^{(i+2,j+1)} + v^{(i+2,j-1)} +
\]

\[+ hf(x^{(i)}, y^{(j)}), \frac{1}{2} [v^{(i+2,j+1)} + v^{(i+2,j-1)}], (u_{hk}(x^{(i)}, y^{(j)}), \Delta_1 v^{(i,j)}), j \in Z.\]
(4) \[ u_{hk}(x, y) = v^{(i,j-1)} + (x - x^{(i)}) \Delta_0 v^{(i,j-1)} + (y - y^{(j-1)}) \Delta_1 v^{(i,j)} + + (x - x^{(i)}) (y - y^{(j-1)}) \Delta_0 \Delta_1 v^{(i,j)}, \text{ where } (x, y) \in P_{ij}, \ j \in 2Z. \]

It is easy to see that (4) defines a continuous function on \([-b_0, a_0] \times R\). In the sequel we will write \((u_{hk})_{(i,j)}\) instead of \((u_{hk})_{(x^{(i)}, y^{(j)})}\).

3. Properties of a Solution of the Difference Equation

**Lemma 1.** If \(f \in C([0, a_0] \times R \times R \times C(B, R) \times R, R)\), and conditions 1°, 4° of Assumption H are satisfied, then for all \(i = 0, \ldots, n_0, j \in Z\) we have

(5) \[ |v^{(i,j)}| \leq N. \]

**Proof.** It follows from (3) that for \(i = 0, \ldots, n_0 - 1, j \in Z\) we have

\[
|v^{(i+1,j)}| \leq |\frac{1}{2}v^{(i,j+1)} + v^{(i,j-1)}| + + h f(x^{(i)}, y^{(i)}), \frac{1}{2}(v^{(i+1,j)} + v^{(i,-j-1)}), (u_{hk})_{(i,j)}, \Delta_1 v^{(i,j)}|.
\]

Let \(\bar{v}^{(i)} = \sup \{|v^{(r,j)}|: -n_1 \leq r \leq i, j \in Z\}\). The boundness of \(\varphi\) implies that \(\bar{v}^{(i)} < +\infty\). From the condition 4° of Assumption H we have \(\bar{v}^{(i+1)} \leq \bar{v}^{(i)} + + h V(\bar{v}^{(i)}), \) and hence

(6) \[
\frac{1}{h} \left[\bar{v}^{(i+1)} - \bar{v}^{(i)}\right] \leq V(\bar{v}^{(i)}), \quad i = 0, \ldots, n_0 - 1.
\]

Let us consider the Cauchy problem

(7) \[ D_x w(x) = V(w(x)), \quad w(0) = \bar{M}. \]

If \(w\) is a solution of (7), then it is a nondecreasing function and hence \(D_x w\) is a composition of two nondecreasing functions. Thus \(w\) is a convex function and from (6) we see that \(\bar{v}^{(i)} \leq w(x^{(i)}), i = 0, \ldots, n_0\). We also have that \(w\) satisfies

\[
\int_{s}^{w(x)} \frac{dt}{V(t)} = x.
\]

From the condition 4° of Assumption H we have then \(w(x) \leq N, x \in [0, a_0]\). Therefore we have \(\bar{v}^{(i)} \leq N, i = 0, \ldots, n_0\), which is equivalent to (5).

**Lemma 2.** If \(f: [0, a_0] \times R \times R \times C(B, R) \times R \to R\) is of class \(C^1\), \((h, k) \in U\) and conditions 1°, 4°, 5° of Assumption H are satisfied, then for all \(i = 0, \ldots, n_0, j \in Z\) we have

(8) \[ \left|\Delta_1 v^{(i,j)}\right| \leq N_1. \]

**Proof.** It follows from (3) that for \(i = 0, \ldots, n_0 - 1, j \in Z\) we have

\[
\Delta_1 v^{(i+1,j)} = \frac{1}{2} \left[\Delta_1 v^{(i,j+1)} + \Delta_1 v^{(i,j-1)}\right] + + h \left[ f(x^{(i)}, y^{(j+1)}), \frac{1}{2}(v^{(i,j+2)} + v^{(i,j)}), (u_{hk})_{(i,j+1)}, \Delta_1 v^{(i,j+1)}\right] - - f(x^{(i)}, y^{(j-1)}), \frac{1}{2}(v^{(i,j)} + v^{(i,j-2)}), (u_{hk})_{(i,j-1)}, \Delta_1 v^{(i,j-1)}\right].
\]

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Using the Lagrange theorem we obtain
\[
\Delta_t v^{(i+1,j)} = \Delta_t v^{(i,j+1)} \left[ \frac{1}{2} + \frac{h}{2k} D_q f(P(i,j)) \right] + \\
+ \Delta_t v^{(i,j-1)} \left[ \frac{1}{2} - \frac{h}{2k} D_q f(P(i,j)) \right] + \\
+ h D_q f(P(i,j)) + \frac{h}{2} (\Delta_t v^{(i,j+1)} + \Delta_t v^{(i,j-1)}) D_q f(P(i,j)) + \\
+ h D_w f(P(i,j)) (r_{i,j}),
\]
where \(P^{(i,j)}\) is an intermediate point, and \(r_{i,j}\) is defined by
\[
r_{i,j} = \frac{1}{2k} \left[ (u_{kk})_{(i,j+1)} - (u_{kk})_{(i,j-1)} \right].
\]
Let \(z^{(i)} = \sup \{ |\Delta_t v^{(r,J)}| : -n_1 \leq r \leq i, j \in Z \}. \) From (5) it follows that \(z^{(i)} < +\infty.\) Since \(|D_q f(P(i,j))| \leq A\) and \((h, k) \in U,\) we have
\[
\left[ \Delta_t v^{(i,j+1)} \left[ \frac{1}{2} + \frac{h}{2k} D_q f(P(i,j)) \right] + \Delta_t v^{(i,j-1)} \left[ \frac{1}{2} - \frac{h}{2k} D_q f(P(i,j)) \right] \right] \leq z^{(i)}.
\]
Thus the above inequality, (5), the condition 5° of Assumption H and \(\|r_{i,j}\|_0 \leq z^{(i)}\) yield
\[
z^{(i+1)} \leq (1 + 2h W(N + 3z^{(i)})) z^{(i)} + h W(N + 3z^{(i)}),
\]
and hence
\[
\frac{1}{h} \left[ z^{(i+1)} - z^{(i)} \right] \leq (2z^{(i)} + 1) W(N + 3z^{(i)}), \quad i = 0, \ldots, n_0 - 1.
\]
Taking into consideration the Cauchy problem
\[
D_x w(x) = (2w(x) + 1) W(N + 3 w(x)), \quad w(0) = \bar{L},
\]
and using the same arguments as in the proof of Lemma 1 we obtain (8).

**Lemma 3.** Suppose that Assumption H is satisfied and that \((h, k) \in U.\) Then for sufficiently small \(k, h\) there is a constant \(N_2 \in R_+\) such that
\[
\Delta^2_t v^{(i,j)} \leq \frac{N_2}{x^{(i)}}, \quad \text{for} \quad b_0 = 0, \quad i = 1, \ldots, n_0, \quad j \in Z,
\]
\[
\Delta^2_t v^{(i,j)} \leq N_2, \quad \text{for} \quad b_0 > 0, \quad i = 0, \ldots, n_0, \quad j \in Z.
\]

**Proof.** We will first prove (9). From (3) it follows that for \(i = 0, \ldots, n_0 - 1,\) \(j \in Z\) we have
\[
\Delta^2_t v^{(i+1,j)} = \frac{1}{2} (\Delta^2_t v^{(i,j+1)} + \Delta^2_t v^{(i,j-1)}) + \\
+ \frac{h}{(2k)^2} \left[ f(x^{(i)}, y^{(j+2)}, v^{(i,j+1)} + v^{(i,j-1)}), (u_{kk})_{(i,j+2)}, \Delta_t v^{(i,j+2)} \right] + \\
- 2f(x^{(i)}, y^{(j)}, \frac{1}{2} (v^{(i,j+1)} + v^{(i,j-1)}), (u_{kk})_{(i,j)}, \Delta_t v^{(i,j)}) + \\
+ f(x^{(i)}, y^{(j-2)}, \frac{1}{2} (v^{(i,j-1)} + v^{(i,j-3)}), (u_{kk})_{(i,j-2)}, \Delta_t v^{(i,j-2)})].
\]

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Let $r_{i,j}$ have the same meaning as in the proof of Lemma 2 and let

$$q_{i,j} = \frac{1}{(2k)^2} \left[ (u_{kh})_{(i,j+2)} - 2(u_{hk})_{(i,j)} + (u_{kh})_{(i,j-2)} \right],$$

$$Q^{(i,j)} = (x^{(i)}, y^{(j)}, 1/2(v^{(i,j+1)} + v^{(i,j-1)}), (u_{kh})_{(i,j)}, \Delta_1 v^{(i,j)}).$$

From the relations

$$v^{(i,j'+1)} + v^{(i,j'-1)} - (v^{(i,j'-1)} + v^{(i,j'-3)}) = 2k(\Delta_1 v^{(i,j')} + \Delta_1 v^{(i,j'-2)}),$$

$$\Delta_1 v^{(i,j'+1)} - \Delta_1 v^{(i,j'-1)} = 2k \Delta_1^2 v^{(i,j')},$$

and from Taylor’s formula we obtain

$$\Delta_1^2 v^{(i,j+1)} = \Delta_1^2 v^{(i,j+1)} + \frac{1}{2} + \frac{h}{2k} D_q f(Q^{(i,j)}) + h D_{qf} f(Q^{(i,j)}) +$$

$$+ \frac{h}{2} \left( \Delta_1 v^{(i,j+2)} + \Delta_1 v^{(i,j)} \right) D_{q^2} f(Q^{(i,j)}) + h D_{qf} f(Q^{(i,j)}) \left( r_{i,j+1} \right) +$$

$$+ \Delta_1^2 v^{(i,j-1)} \left[ \frac{1}{2} + \frac{h}{2k} D_q f(Q^{(i,j)}) + h D_{qf} f(Q^{(i,j)}) \right] +$$

$$+ \frac{h}{2} \left( \Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j-2)} \right) D_{q^2} f(Q^{(i,j)}) + h D_{qf} f(Q^{(i,j)}) \left( r_{i,j-1} \right) +$$

$$+ \frac{h}{2} \left( \Delta_1^2 v^{(i,j+1)} + \Delta_1^2 v^{(i,j-1)} \right) D_p f(Q^{(i,j)}) + h D_{p} f(Q^{(i,j)}) \left( q_{i,j} \right) + R^{(i,j)},$$

where $Q^{(i,j)}$, $Q^{(i,j)}_2$ are intermediate points and

$$R^{(i,j)} = \frac{h}{2} \left[ D_{q^2} f(Q^{(i,j)}) + D_{qf} f(Q^{(i,j)}) \right] +$$

$$+ \frac{h}{2} \left[ \frac{1}{2} \Delta_1 v^{(i,j+2)} + \Delta_1 v^{(i,j)} \right] D_{p^2} f(Q^{(i,j)}) +$$

$$+ \left( \Delta_1 v^{(i,j+2)} + \Delta_1 v^{(i,j)} \right) D_{p^2} f(Q^{(i,j)}) \left( r_{i,j+1} \right) +$$

$$+ \frac{h}{2} \left[ \frac{1}{2} \Delta_1 v^{(i,j+2)} + \Delta_1 v^{(i,j-2)} \right] D_{p^2} f(Q^{(i,j)}) +$$

$$+ h \left[ \frac{1}{2} D_{ww} f(Q^{(i,j)}) \left( r_{i,j+1} \right) + D_{ww} f(Q^{(i,j)}) \left( r_{i,j-1} \right) +$$

$$+ \left( \Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j+2)} \right) D_{pp} f(Q^{(i,j)}) \left( r_{i,j+1} \right) +$$

$$+ h \left[ \frac{1}{2} D_{ww} f(Q^{(i,j)}) \left( r_{i,j-1} \right) + D_{ww} f(Q^{(i,j)}) \left( r_{i,j-1} \right) +$$

$$+ \left( \Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j-2)} \right) D_{pp} f(Q^{(i,j)}) \left( r_{i,j-1} \right) +$$

$$+ \frac{h}{2} \left[ \Delta_1^2 v^{(i,j+1)} \right] D_{q^2} f(Q^{(i,j)}) + h \left[ \Delta_1^2 v^{(i,j-1)} \right] D_{q^2} f(Q^{(i,j)}) \right).$$

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We will first estimate $\Delta^2_{11}v^{(i,j)}$ for $i$ such that $\chi^{(i)} \leq \delta$, where $\delta$ is the constant from the condition $8^\circ$ from Assumption H. For $i = 0, \ldots, n_0 - 1$, $j \in \mathbb{Z}$ we write

\[
\begin{align*}
A^{(i,j)}_1 &= D^2_{qq}f(Q^{(i,j)}_1) + \frac{1}{2}(\Delta_1v^{(i-j+2)} + \Delta_1v^{(i-j)}) \cdot D^2_{qq}f(Q^{(i,j)}_1) + \\
&+ D^2_{qq}f(Q^{(i,j)}_1)(r_{i,j+1}) + \frac{1}{2}D_{pp}f(Q^{(i,j)}_1), \\
A^{(i,j)}_2 &= D^2_{qq}f(Q^{(i,j)}_2) + \frac{1}{2}(\Delta_1v^{(i-j)} + \Delta_1v^{(i-j-2)}) \cdot D^2_{qq}f(Q^{(i,j)}_2) + \\
&+ D^2_{qq}f(Q^{(i,j)}_2)(r_{i,j-1}) + \frac{1}{2}D_{pp}f(Q^{(i,j)}_2).
\end{align*}
\]

There are constants $d_1, d_2 \in \mathbb{R}^+$, $d_1 > d_2$, and

\[
c = \min\left(\frac{\mu}{2}, \frac{A(1 + d_1h)}{2N_1}\right)
\]

such that

\[
A^{(i,j)}_1 + A^{(i,j)}_2 \leq d_1, \quad R^{(i,j)} \leq d_2h - c[\Delta^2_{11}v^{(i,j+1)} + \Delta^2_{11}v^{(i,j-1)}]h.
\]

Let $h, k$ be sufficiently small so that

\[
\frac{1}{2} + \frac{h}{2k}D_{qq}f(Q^{(i,j)}) + hA^{(i,j)}_1 \geq 0, \quad \frac{1}{2} - \frac{h}{2k}D_{qq}f(Q^{(i,j)}) + hA^{(i,j)}_2 \geq 0.
\]

We introduce the following notations

\[
S^{(i,j)} = \max\{\Delta^2_{11}v^{(i,j+1)}, \Delta^2_{11}v^{(i,j-1)}, 0\},
\]

\[
\bar{S}^{(i)} = \sup\{S^{(\tau,j)}: 0 \leq \tau \leq i, j \in \mathbb{Z}\}.
\]

From (5) it follows that $\bar{S}^{(i)} < +\infty$. Therefore we have

\[
\Delta^2_{11}v^{(i+1,j)} \leq S^{(i,j)}[1 + d_1h] + \\
+ W(N + 3N_1)\bar{S}^{(i)}h + d_2h - c(S^{(i,j)})^2h.
\]

From the inequality

\[
\Delta^2_{11}v^{(i,j)} = \frac{1}{2k}((\Delta_1v^{(i,j+1)} - \Delta_1v^{(i,j-1)}) \leq \frac{N_1}{k},
\]

and from conditions $A < k/h$ and $c \leq A(1 + d_1h)/2N_1$ we obtain

\[
\Delta^2_{11}v^{(i,j)} \leq \frac{1 + d_1h}{2ch}.
\]

Let us consider the polynomial $H(y) = y(1 + d_1h) + W(N + 3N_1)\bar{S}^{(i)}h + d_2h - chy^2$. It is easy to see that $D_y H(y) = 1 + d_1h - 2chy \geq 0$ for $y \leq (1 + d_1h)/2ch$.

By force of (11) we derive then

\[
\Delta^2_{11}v^{(i+1,j)} \leq \bar{S}^{(i)}[1 + (d_1 + W(N + 3N_1))h] + d_2h - c(\bar{S}^{(i)})^2h.
\]

The right hand side of the above inequality is positive, which gives

\[
\bar{S}^{(i+1)} \leq \bar{S}^{(i)}[1 + (d_1 + W(N + 3N_1))h] + d_2h - c(\bar{S}^{(i)})^2h.
\]
Hence for \( \tilde{S}^{(i)} = \tilde{S}^{(i)} + 1 \) we have
\[
\tilde{S}^{(i+1)} \leq \tilde{S}^{(i)}[1 + (d_1 + W(N + 3N_1) + 2c) h] +
+ d_2 h - (d_1 + W(N + 3N_1) + c) h - c(\tilde{S}^{(i)})^2 h.
\]
Using the condition \( d_1 > d_2 \) and putting \( \tilde{d} = d_1 + W(N + 3N_1) + 2c \) we obtain
\[
\tilde{S}^{(i+1)} \leq \tilde{S}^{(i)}(1 + \tilde{d} h) - c(\tilde{S}^{(i)})^2 h.
\]
Let \( h \) be so small that \( \tilde{d} h \leq \frac{1}{2} \). Then multiplying the last inequality by \( (1 - \tilde{d} h)^{i+1} \) and putting \( W^{(i)} = (1 - \tilde{d} h)^i \tilde{S}^{(i)} \) we get
\[
W^{(i+1)} \leq W^{(i)} - c h W^{(i)} (1 - \tilde{d} h)^{-i+1} \leq W^{(i)} - c h W^{(i)}^2, \quad i > 0,
\]
\[
W^{(i)} \leq W^{(0)} - c h W^{(0)}^2 (1 - \tilde{d} h) \leq W^{(0)} - \frac{c}{2} W^{(0)}^2,
\]
and hence
\[
\frac{1}{h} [W^{(i+1)} - W^{(i)}] \leq -\frac{c}{2} W^{(i)}.
\]
Let us consider the following Cauchy problem
\[
D_x w(x) = -\frac{c}{2} w^2(x), \quad w(0) = W^{(0)}.
\]
The solution of this problem is given by
\[
w(x) = \frac{1}{\frac{c}{2} x + \frac{1}{W^{(0)}}}.
\]
We see that \( w \) is convex and then by force of (12) we have
\[
W^{(i)} \leq w(x^{(i)}) = \frac{1}{\frac{c}{2} x^{(i)} + \frac{1}{W^{(0)}}} < \frac{2}{c x^{(i)}}, \quad i > 0.
\]
From the above inequality we derive
\[
(1 - \tilde{d} h)^i (\tilde{S}^{(i)} + 1) \leq \frac{2}{c x^{(i)}},
\]
and hence from the condition \( \tilde{d} h \leq \frac{1}{2} \) we have
\[
\tilde{S}^{(i)} \leq \frac{1}{x^{(i)}} \left[ \frac{2}{c} (1 - \tilde{d} h)^{-i} - i h \right] \leq \frac{1}{x^{(i)}} \left\{ \frac{2}{c} \left[ (1 - \tilde{d} h)^{-1/\tilde{d} h} \right] \tilde{d} h - i h \right\} \leq \frac{\tilde{N}_2}{x^{(i)}},
\]
where
\[
\tilde{N}_2 = \frac{2}{c} \exp (\tilde{d} \delta \ln 4) - \delta.
\]
For \( i \) such that \( x^{(i)} > \delta \) we analogously derive
\[
S^{(i+1)} \leq S^{(i)}[1 + (d_1 + W(N + 3N_1))h] + d_2h.
\]
By force of the mathematical induction we have
\[
S^{(i)} \leq \exp(d_a_0) S^{(0)} + \frac{d_2}{\hat{d}} \left[ \exp(d_a_0) - 1 \right],
\]
where \( \hat{d} = d_1 + W(N + 3N_1) \) and \( \lceil t \rceil \) denotes the integral part of \( t \). Supposing that \( h \) is so small that \( h \leq \delta/2 \) and putting
\[
N_2 = \max \left\{ N_2, \frac{2N_2}{\delta} \exp(d_a_0) + \frac{d_2}{\hat{d}} \left[ \exp(d_a_0) - 1 \right] \right\}
\]
we get (9).

In order to prove (10), let us adopt the previous notations \( r_{i,j}, q_{i,j}, Q^{(i,j)}, Q_1^{(i,j)}, Q_2^{(i,j)}, R^{(i,j)}, A_1^{(i,j)}, A_2^{(i,j)} \) and let \( h, k \) be so small that
\[
\frac{1}{2} + \frac{h}{2k} D_q f(Q^{(i,j)}) + hA_1^{(i,j)} \geq 0, \quad \frac{1}{2} - \frac{h}{2k} D_q f(Q^{(i,j)}) + hA_2^{(i,j)} \geq 0.
\]
There are constants \( d_1, d_2 \) such that
\[
A_1^{(i,j)} + A_2^{(i,j)} \leq d_1, \quad R^{(i,j)} \leq d_2h.
\]
Using the previous arguments we prove that
\[
S^{(i+1)} \leq S^{(i)}[1 + (d_1 + W(N + 3N_1))h] + d_2h, \quad i = 0, \ldots, n_0 - 1,
\]
where \( S^{(i)} = \sup \{ A_1^{(i)} (v^{(i,j)}): -n_1 \leq \tau \leq i, j \in Z \} \). Now, analogously like (13) we obtain (10) with
\[
N_2 = \left( \bar{K} + \frac{d_2}{\hat{d}} \right) \exp(d_a_0) - \frac{d_2}{\hat{d}},
\]
where \( \bar{K} \) is the constant from the condition 2\(^o\) of Assumption \( H \). This ends the proof.

Let \( B \) be some constant such that \( B > A \). By \( \bar{U} \) we define the set \( \{(h, k): A < k/h \leq B\} \).

**Lemma 4.** Suppose that Assumption \( H \) is satisfied, \( (h, k) \in \bar{U}, z, Y > 0, 0 \leq i \leq n_0 \) and additionally \( i \geq z/h \) in the case \( b_0 = 0 \). Then there is a constant \( C = C(Y,z) \) for \( b_0 = 0 \), or \( C = C(Y) \) for \( b_0 > 0 \) such that
\[
\sum_{|j| \leq Y/k} 2k|\Delta_1^{(j)} v^{(i,j)}| \leq C.
\]

**Proof.** Let \( b_0 = 0, \bar{A} \geq N_2/z \) and \( u^{(i,j)} = \Delta_1 v^{(i,j)} - \bar{A}k \). Using (9) for any \( i > z/h, j \in Z \) we have
\[
u^{(i,j+1)} - u^{(i,j-1)} = \Delta_1 v^{(i,j+1)} - \Delta_1 v^{(i,j-1)} - 2\bar{A}k = 2k(\Delta_1^2 v^{(i,j)} - \bar{A}) \leq 2k \left( \frac{N_2}{ih} - \bar{A} \right) \leq 0.
\]

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If $b_0 > 0$, then we take $\bar{A} \geq N_2$ and by force of (10) we obtain the same estimate for $i = 0, \ldots, n_0, j \in Z$. From this we have
\[
\sum_{|j| \leq Y/k} 2k|\Delta_1 u^{(i,j)}| = \sum_{|j| \leq Y/k} [u^{(i,j+1)} - u^{(i,j-1)}] \leq
\]
\[
\leq 4 \max_{|j| \leq Y/k} |u^{(i,j)}| \leq 4(N_1 + \bar{A} Y),
\]
and
\[
\sum_{|j| \leq Y/k} 2k|\Delta_1^2 v^{(i,j)}| = \sum_{|j| \leq Y/k} |\Delta_1 v^{(i,j+1)} - \Delta_1 v^{(i,j-1)}| =
\]
\[
= \sum_{|j| \leq Y/k} |u^{(i,j+1)} - u^{(i,j-1)} + 2\bar{A} k| \leq \sum_{|j| \leq Y/k} 2k|\Delta_1 u^{(i,j)}| +
\]
\[
+ 4\bar{A} Y + 2\bar{A} B a_0.
\]
Thus (14) is satisfied with $C = 4N_1 + 8\bar{A} Y + 2\bar{A} B a_0$.

Remark 2. The analogous properties to that proved in Lemmas 2–4 we may obtain also for the operator $\bar{A}_1$ defined by $\bar{A}_1 v^{(i,j)} = (1/k) [v^{(i,j)} - v^{(i,j-1)}]$.

For any $1 \leq i \leq n_0$, $j \in Z$, $0 \leq n \leq i - 1$ let $U_{ij}(n) = \{ s \in Z: s = j + i - n \in 2Z \}$.

Lemma 5. If $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$ is of class $C^1$, $(h, k) \in U$ and conditions 1°, 4°, 5° of Assumption H are satisfied, then there are constants $a_{i,j}^{n,s} \geq 0$, $\eta_{i,j}^n$, $i = 1, \ldots, n_0$, $j \in Z$, $n = 0, \ldots, i - 1$, $s \in U_{i,j}(n)$, $j - (i - n) \leq s \leq j + (i - n)$, such that
\[
\Delta_1 v^{(i,j)} = \sum_{s = j - (i - n)}^{j + (i - n)} a_{i,j}^{n,s} \Delta_1 v^{(n,s)} + \eta_{i,j}^n,
\]
\[
\sum_{s = j - (i - n)}^{j + (i - n)} a_{i,j}^{n,s} = 1,
\]
\[
|\eta_{i,j}^n| \leq (i - n) h c_1,
\]
where $c_1 = W(N + 3N_1)(1 + 2N_1)$.

Proof. Analogously like in Lemma 2 we get
\[
\Delta_1 v^{(i,j)} = \Delta_1 v^{(i-1,j+1)} \left[ \frac{1}{2} + \frac{h}{2k} D_q f(P^{(i-1,j)}) \right] +
\]
\[
+ \Delta_1 v^{(i-1,j-1)} \left[ \frac{1}{2} - \frac{h}{2k} D_q f(P^{(i-1,j)}) \right] + \eta_{i,j}^{i-1},
\]
where
\[
\eta_{i,j}^{i-1} = h D_p f(P^{(i-1,j)}) +
\]
\[
+ \frac{h}{2} (\Delta_1 v^{(i-1,j+1)} + \Delta_1 v^{(i-1,j-1)}) D_p f(P^{(i-1,j)}) +
\]
\[
+ h D_w f(P^{(i-1,j)})(r_{i-1,j}), \quad |\eta_{i,j}^{i-1}| \leq h c_1.
\]
Thus the lemma holds for $n = i - 1$. Assume that the lemma holds for some $n < i$,
we will prove it for \( n - 1 \). For \( i = 1, \ldots, n_0, j \in Z \) we have

\[
\Delta_t v^{(i,j)} = \sum_{s=j-(i-n)}^{j+(i-n)} a^n_{i,j}^s \Delta_t v^{(n,s)} + \eta^n_{i,j} =
\]

\[
= \sum_{s=j-(i-n)}^{j+(i-n)} a^n_{i,j}^s \left\{ \Delta_t v^{(n-1,s+1)} \left[ \frac{1}{2} + \frac{h}{2k} D_q f(P^{(n-1,s+1)}) \right] + \right. \\
+ \Delta_t v^{(n-1,s-1)} \left[ \frac{1}{2} - \frac{h}{2k} D_q f(P^{(n-1,s-1)}) \right] + \eta^n_i \left. \right\} + \eta^n_{i,j},
\]

where

\[
\eta^n_i = h D_q f(P^{(n-1,s)}) + \frac{h}{2} \left( \Delta_t v^{(n-1,s+1)} + \Delta_t v^{(n-1,s-1)} \right) D_q f(P^{(n-1,s)}) + \\
+ h D_w f(P^{(n-1,s)}) (r^{(n-1,s)}).
\]

We define constants \( a^n_{i,j}^{-1,s} \), \( \eta^n_{i,j}^{-1} \) in the following way:

\[
a^n_{i,j}^{-1,s} = a^n_{i,j}^{s-1} \left[ \frac{1}{2} + \frac{h}{2k} D_q f(P^{(n-1,s-1)}) \right] + \\
+ a^n_{i,j}^{s+1} \left[ \frac{1}{2} - \frac{h}{2k} D_q f(P^{(n-1,s+1)}) \right],
\]

\( s = j - (i-n) + 1, \ldots, j + (i-n) - 1, \)

\[
a^n_{i,j}^{-1,j-(i-n)-1} = a^n_{i,j}^{-1,j-(i-n)} \left[ \frac{1}{2} + \frac{h}{2k} D_q f(P^{(n-1,j-(i-n))}) \right],
\]

\[
a^n_{i,j}^{-1,j+(i-n)+1} = a^n_{i,j}^{-1,j+(i-n)} \left[ \frac{1}{2} - \frac{h}{2k} D_q f(P^{(n-1,j+(i-n))}) \right],
\]

\[
\eta^n_{i,j}^{-1} = \sum_{s=j-(i-n)}^{j+(i-n)} a^n_{i,j}^s \eta^n_i + \eta^n_{i,j}.
\]

For these constants we have

\[
\Delta_t v^{(i,j)} = \sum_{s=j-(i-n)-1}^{j+(i-n)+1} a^n_{i,j}^{-1,s} \Delta_t v^{(n-1,s)} + \eta^n_{i,j}^{-1},
\]

\[
\sum_{s=j-(i-n)-1}^{j+(i-n)+1} a^n_{i,j}^{-1,s} = 1,
\]

\[
|\eta^n_{i,j}^{-1}| \leq (i - n + 1) h c_1,
\]

which completes the proof of Lemma 5.

**Lemma 6.** Suppose that Assumption H is satisfied, \((h, k) \in \mathcal{U}, \alpha, Y > 0, 1 \leq i \leq n_0, 0 \leq n \leq i - 1\) and additionally \( i > \alpha/h, n > \alpha/h \) in the case \( b_0 = 0 \). Then
there is a constant \( L = L(Y, \alpha) \) for \( b_0 = 0 \), or \( L = L(Y) \) for \( b_0 > 0 \) such that

\[
\sum_{|J| \leq Y/k} 2k|\Delta^a v^{(i,j)} - \Delta^a v^{(n,j)}| \leq Lh(i - n).
\]

**Proof.** We will first prove (15) for the case \( i - n \in 2Z \). By force of Lemma 5 we have

\[
\sum_{|J| \leq Y/k} 2k|\Delta^a v^{(i,j)} - \Delta^a v^{(n,j)}| \leq \\
\leq \sum_{|J| \leq Y/k} \sum_{s = j - (i-n)}^{j + (i-n)} a^s_{i,j} \sum_{x \in V_{i,j}(n)} |\Delta^a v^{(n,s)} - \Delta^a v^{(n,j)}| 2k + \\
+ (4Y + 2Ba_0) c_1(i - n) h \leq \\
\leq \sum_{|J| \leq Y/k} \sum_{s = j - (i-n)}^{j + (i-n)} a^s_{i,j} \sum_{x \in V_{i,j}(n)} |\Delta^a v^{(n,s+1)} - \Delta^a v^{(n,s-1)}| 2k + \\
+ (4Y + 2Ba_0) c_1(i - n) h \leq \\
\leq [2(i - n) + 1] 2k \sum_{|J| \leq Y/k} \sum_{s = j - (i-n)}^{j + (i-n)} |\Delta^a v^{(n,s+1)} - \Delta^a v^{(n,s-1)}| + \\
+ (4Y + 2Ba_0) c_1(i - n) h.
\]

Using (14) with \( C = C(Y + Ba_0, \alpha) \) for \( b_0 = 0 \), or \( C = C(Y + Ba_0) \) for \( b_0 > 0 \) we get (15) with the constant \( L = 6BC + 4Yc_1 + 2Ba_0c_1 \). From Remark 2 we obtain (15) for \( i - n \notin 2Z \).

**4. THE SEQUENCE OF APPROXIMATE SOLUTIONS**

**Lemma 7.** Suppose that \( f : [0, a_0] \times R \times R \times C(B, R) \times R \to R \) is of class \( C^1 \) and that conditions 1', 4', 5' of Assumption H are satisfied. Then there is a sequence \( \{ (h_v, k_v) \}_{v=1}^{\infty} \), \( (h_v, k_v) \in \bar{U} \), \( \lim_{v \to \infty} h_v = \lim_{v \to \infty} k_v = 0 \), and a function \( \bar{u} \in C([0, a_0] \times R, R) \) such that \( \lim_{v \to \infty} u_{h_v,k_v}(x, y) = \bar{u}(x, y) \) almost uniformly on \([0, a_0] \times R\).

This lemma follows from Lemmas 1, 2 and from Remark 2.

Let us define sequences \( \{u^{(v)}\}_{v=1}^{\infty}, \{V^{(v)}\}_{v=1}^{\infty}, \{W^{(v)}\}_{v=1}^{\infty} \). We put \( u^{(v)} = u_{h_v,k_v} \). If \( (x, y) \in [0, a_0] \times R \), then there are \( i, j, 0 \leq i \leq n_0 - 1, j \in 2Z \) such that \( (x, y) \in [x^{(i)}, x^{(i+1)}] \times [y^{(j-1)}, y^{(j+1)}] \). Let

\[
V^{(v)}(x, y) = \Delta^a v^{(i,j)} + (x - x^{(i)}) \Delta^2 v^{(i,j)},
\]

\[
W^{(v)}(x, y) = \Delta^a v^{(i,j-1)} + (y - y^{(j-1)}) \Delta^2 v^{(i,j)},
\]

where the difference operators are defined for \( h = h_v, k = k_v \). If \( i = n_0 - 1 \), then we replace in the above definitions the interval \([x^{(i)}, x^{(i+1)}]\) by \([x^{(i)}, x^{(i+1)}]\). Thus we have \( V^{(v)}, W^{(v)} : [0, a_0] \times R \to R \) and \( V^{(v)}(x, y) = D_x u^{(v)}(x, y), W^{(v)}(x, y) = D_x u^{(v)}(x, y) \) a.e. on \([0, a_0] \times R\).
By $\text{Var}_v[V^{(v)}(x, \cdot)]$ we denote the variation of the function $V^{(v)}(x, \cdot)$ on $[-Y, Y]$, $Y > 0$.

**Lemma 8.** If Assumption H is satisfied and $\alpha$, $Y > 0$, then for each $x \in [\alpha, a_0]$, $(x \in [0, a_0]$ in the case $b_0 > 0)$ and for any integer $v$ we have $\text{Var}_v[V^{(v)}(x, \cdot)] \leq 3C$, where $C$ is the constant from (14).

**Proof.** For any $x \in [0, a_0]$ there is $i_0 \leq i \leq n_0 - 1$ such that $x \in [x^{i_0}, x^{i_0+1})$. Since $V^{(v)}(x, \cdot)$ is a constant function on intervals $[y^{j-i}, y^{j+i+1})$, $j \in \mathbb{Z}$, we have

$$
\text{Var}_v[V^{(v)}(x, \cdot)] = \sum_{j \in \mathbb{Z}} \left| V^{(v)}(x, y^{j+i+1}) - V^{(v)}(x, y^{j-i-1}) \right| \leq \sum_{|j| \leq Y/k_v} \left| V^{(v)}(x, y^{j+i+1}) - V^{(v)}(x, y^{j-i-1}) \right| \leq \\
\sum_{|j| \leq Y/k_v} \frac{|\Delta_1 v^{(j+i+1)} - \Delta_1 v^{(j-i-1)}|}{h_v} + \left( x - x^{(j)} \right) \Delta_0 v^{(j+i+1)} - \Delta_1 v^{(j-i+1)} - \Delta_1 v^{(j-i-1)} = \sum_{|j| \leq Y/k_v} \left| \Delta_1 v^{(j+i+1)} - \Delta_1 v^{(j-i+1)} - \Delta_1 v^{(j-i-1)} \right| \\
\leq \sum_{|j| \leq Y/k_v} 2k_v |\Delta_1 v^{(j+i+1)} - \Delta_1 v^{(j-i+1)}| + \frac{x - x^{(j)} |\Delta_1 v^{(j+i+1)} - \Delta_1 v^{(j-i+1)} - \Delta_1 v^{(j-i-1)}|}{h_v} \leq \\
\sum_{|j| \leq Y/k_v} \frac{2k_v |\Delta_1 v^{(j+i+1)} - \Delta_1 v^{(j-i+1)}|}{h_v} + \frac{x - x^{(j)} (2k_v |\Delta_1 v^{(j+i+1)}| + 2k_v |\Delta_1 v^{(j-i+1)}|)}{h_v}.
$$

The above inequality and (14) complete the proof of Lemma 8.

By $L(Y)$ we denote a set of all Lebesgue integrable functions $\psi: [0, a_0] \times \times [-Y, Y] \to R$ with the norm $\|\psi\|_{L(Y)} = \int_0^{a_0} \int_Y^Y |\psi(x, y)| \, dy$. $\psi(x, y)$

**Lemma 9.** If Assumption H is satisfied, then there is a sequence $\{v_s\}_{s=1}^{\infty}$ and a measurable function $\tilde{v}: [0, a_0] \times R \to R$ such that $\lim_{s \to \infty} \|V^{(v_s)} - \tilde{v}\|_{L(Y)} = 0$.

**Proof.** Let $Y > 0$, $\alpha \in (0, a_0)$ and let $\{x_r\}_{r=1}^{\infty}$ be a sequence of all rational numbers from the interval $[\alpha, a_0]$. It is easy to see that for any integer $v$ we have $|V^{(v)}(x, y)| \leq 3N_1$, $(x, y) \in [0, a_0] \times R$. From this and from Lemma 8 it follows that assumptions of Helly’s theorem are satisfied. Hence for any $x \in [\alpha, a_0]$ there is a subsequence of the sequence $\{V^{(v)}(x, \cdot)\}_{v=1}^\infty$ which is convergent on $[-Y, Y]$. If we apply the diagonal process, then we obtain a subsequence of $\{V^{(v)}(x, \cdot)\}_{v=1}^\infty$, which is convergent on the set $\{(x, y) \in [\alpha, a_0] \times R: x = x_r \text{ for some } r\}$. We denote this sequence by $\{V^{(v_s)}(x_r, \cdot)\}_{s=1}^\infty$ again.

We will prove that

$$
\lim_{v, s \to \infty} \int_Y^Y |V^{(v)}(x, y) - V^{(v_s)}(x, y)| \, dy = 0,
$$

uniformly with respect to $x \in [\alpha, a_0]$.

For each $\varepsilon > 0$ there is a finite subset $\{x_{r_1}, \ldots, x_{r_m}\}$ of the sequence $\{x_r\}_{r=1}^{\infty}$ such that the distance between any two successive elements of this set is less then $\varepsilon/5L$, where $L$ is the constant from inequality (15). For sufficiently large $v, s$ we have

$$
\int_Y^Y |V^{(v)}(x_{r_l}, y) - V^{(v_s)}(x_{r_l}, y)| \, dy < \varepsilon/5, \quad l = 1, \ldots, m.
$$

For each $x \in [\alpha, a_0]$ there is $l_1 \leq l \leq m$ such that $0 \leq x - x_{r_l} < \varepsilon/5L$. For any $v, s$.
we have then
\[ \int_{\Omega} \left| V^{(v)}(x, y) - V^{(v)}(x, y) \right| \, dy \leq \int_{\Omega} \left| V^{(v)}(x, y) - V^{(v)}(x, y) \right| \, dy + \\
+ \int_{\Omega} \left| V^{(v)}(x, y) - V^{(v)}(x, y) \right| \, dy + \\
+ \int_{\Omega} \left| V^{(v)}(x, y) - V^{(v)}(x, y) \right| \, dy. \]

Using the definition of \( V^{(v)} \) we get
\[ \int_{\Omega} \left| V^{(v)}(x, y) - V^{(v)}(x, y) \right| \, dy \leq \\
\leq \sum_{j \geq 2, \ldots, j \leq v} 2k_v \left[ \Delta_t b^{(i,j)} + \frac{x - x^{(i)}}{h_v} \left( \Delta_t b^{(i+1,j)} - \Delta_t b^{(i,j)} \right) \right] - \\
- \left[ \Delta_t b^{(i',j)} + \frac{x_{r_i}}{h_v} - \frac{x^{(i')}}{h_v} \left( \Delta_t b^{(i',1,j)} - \Delta_t b^{(i',j)} \right) \right], \]

where \( i = \lfloor x/h_v \rfloor, i' = \lfloor x_r/h_v \rfloor \). Hence by force of (15) we have
\[ \int_{\Omega} \left| V^{(v)}(x, y) - V^{(v)}(x, y) \right| \, dy \leq \sum_{j \geq 2, \ldots, j \leq v} 2k_v \left[ \left| \Delta_t b^{(i,j)} - \Delta_t b^{(i,j)} \right| + \\
\left| \Delta_t b^{(i+1,j)} - \Delta_t b^{(i,j)} \right| + \left| \Delta_t b^{(i',1,j)} - \Delta_t b^{(i',j)} \right| \right] \leq \\
\leq Lh_v \left( \left[ \frac{x}{h_v} \right] - \left[ \frac{x_{r_i}}{h_v} \right] \right) + 2Lh_v \leq L(x - x_{r_i}) + 3Lh_v < \frac{2}{\delta}c, \]

for \( v \) sufficiently large. Finally, we obtain
\[ \int_{\Omega} \left| V^{(s)}(x, y) - V^{(v)}(x, y) \right| \, dy < \epsilon, \]

for \( s, v \) sufficiently large. This ends the proof of (16).

Since the convergence in (16) is uniform on \([a, a_0]\) for any \( a \in (0, a_0) \) we obtain the almost uniform convergence on \([0, a_0]\). From this we have \( \lim_{v,s \to \infty} \| V^{(s)} - V^{(v)} \|_{L(Y)} = 0. \) The completeness of \( L(Y) \) completes the proof of Lemma 9.

Remark 3. If \( b_0 > 0 \), then it is not necessary to consider the interval \([a, a_0]\), \( a > 0 \), because (16) holds uniformly with respect to \( x \in [0, a_0]. \)

5. THE MAIN THEOREM

Theorem 1. If Assumption H is satisfied, then there is a function \( u \in C([-b_0, a_0] \times R, R) \) which is a generalized solution of (1), (2).

Proof. It follows from Lemma 7 that there is a sequence \( \{(h_v, k_v)\}_{v=1}^{\infty}, (h_v, k_v) \in \tilde{U} \) such that the sequence \( \{u^{(v)}\}_{v=1}^{\infty}, u^{(v)} = u^{(h_v, k_v)} \) is uniformly convergent to a function \( \bar{u} \) on \([0, a_0] \times R\). By force of Lemma 9 there is a subsequence of \( \{V^{(v)}\}_{v=1}^{\infty} \) which is convergent in the \( L(Y) \) norm to \( \bar{e} \). The sequence and its subsequence we denote by the same symbol for simplicity. Let \( \tilde{u}(x, y) = \varphi(x, y) \) for \((x, y) \in [-b_0, 0] \times R\). Then the sequence \( \{\varphi_{h_k}^{(v)}\}_{v=1}^{\infty} \) is uniformly convergent to \( \tilde{u} \) on \([-b_0, 0] \times R\). For
\[(x, y) \in [0, a_0] \times R \text{ we write} \]
\[\tilde{f}^{(v)}(x, y) = f(x, y, u(x, y), u^{(y)}(x, y), V^{(y)}(x, y)), \]
\[\tilde{w}(x, y) = f(x, y, \bar{u}(x, y), \bar{u}^{(y)}(x, y), \bar{v}(x, y)). \]

We will prove that \[\lim_{v \to \infty} \|W^{(v)} - \tilde{w}\|_{L^2} = 0, Y > 0.\] From \[\|W^{(v)} - \tilde{w}\|_{L^2} \leq \|W^{(v)} - \tilde{f}^{(v)}\|_{L^2} + \|\tilde{f}^{(v)} - \tilde{w}\|_{L^2},\]
\[\lim_{v \to \infty} \|\tilde{f}^{(v)} - \tilde{w}\|_{L^2} = 0,\]
we see that it is sufficient if we prove
\[(17) \lim_{v \to \infty} \|W^{(v)} - \tilde{f}^{(v)}\|_{L^2} = 0.\]

If \(x > 0\), then for each \(x \in [x^{(i)}, x^{(i+1)}]\) there is \(i, 0 \leq i \leq n_0 - 1\) such that \(x \in [x^{(i)}, x^{(i+1)}]\). (If \(b_0 > 0\), then we take \(x \in [0, a_0]\)). We have then
\[\int_{Y}^{Y} \left| W^{(v)}(x, y) - \tilde{f}^{(v)}(x, y) \right| dy \leq \]
\[\sum_{j \in Z, |j|, y \leq Y} \int_{x^{(i)}}^{x^{(i+1)}} \left| f(x^{(i)}, y, u^{(v)}(x^{(i)})), \Delta_t v^{(i, j)} \right| dy + \]
\[+ \sum_{j \in Z, |j|, y \leq Y} \frac{y - y^{(j-1)}}{h_y} \left| \Delta_t v^{(i, j)} - \Delta_t v^{(i, j-1)} \right| dy \leq \]
\[\sum_{j \in Z, |j|, y \leq Y} \int_{x^{(i)}}^{x^{(i+1)}} \left| f(x^{(i)}, y^{(j-1)}, v^{(i, j)}, u^{(v)}(x^{(i)})), \Delta_t v^{(i, j-1)} \right| - \]
\[\tilde{f}^{(v)}(x, y) \right| dy + \sum_{j \geq 1, y \leq Y} \frac{2k_y}{h_y} \left| v^{(i, j)} + v^{(i, j-2)} - v^{(i, j-1)} \right| + \]
\[+ \sum_{j \geq 1, y \leq Y} \frac{(2k_y)^2}{h_y} \left| \Delta_t v^{(i, j)} - \Delta_t v^{(i, j-1)} \right|. \]

Using the Lipschitz condition for \(f\), Lemmas 1–3 and Remark 2 we see that the first component of the right hand side of the above inequality tends to zero if \(v \to \infty\). From Remark 2 we obtain that there is a constant \(C_0\) such that
\[\sum_{j \geq 1, y \leq Y} \frac{2k_y}{h_y} \left| v^{(i, j)} + v^{(i, j-2)} - v^{(i, j-1)} \right| \leq BC_0 k_y. \]

Furthermore, from Lemma 6 we have
\[\sum_{j \geq 1, y \leq Y} \frac{(2k_y)^2}{h_y} \left| \Delta_t v^{(i+1, j)} - \Delta_t v^{(i, j)} \right| \leq 2BLh_y. \]

Finally, we obtain
\[\lim_{v \to \infty} \int_{Y}^{Y} \left| W^{(v)}(x, y) - \tilde{f}^{(v)}(x, y) \right| dy = 0, \]
almost uniformly with respect to \(x \in [0, a_0]\), from which we have (17).
From Lemmas 1, 2 and from Remark 2 we obtain that $\tilde{u}$ satisfies the Lipschitz condition on $[0, a_0] \times R$, and hence the derivatives $D_x\tilde{u}, D_y\tilde{u}$ exist a.e. on $[0, a_0] \times \times R$. Since the sequences $\{V^{(v)}_{j}\}_{j=1}^{\infty}$, $\{W^{(v)}_{j}\}_{j=1}^{\infty}$ are equibounded and convergent in the $L(Y)$ norm to $\tilde{u}, \bar{w}$ respectively, it follows that $D_x\tilde{u} = \bar{w}, D_y\tilde{u} = \bar{v}$ a.e. on $[0, a_0] \times \times [-Y, Y]$ for any $Y > 0$. Thus $\tilde{u}$ satisfies (1) a.e. on $[0, a_0] \times R$. From the definition of $\tilde{u}$ we see that the initial condition (2) holds. Furthermore, it is easy to prove the existence of a constant $M > 0$ such that for any $(x, y) \in [0, a_0] \times R, \ l \in R, \ l \neq 0$ we have

$$l^{-2}\left[\tilde{u}(x, y + l) - 2\tilde{u}(x, y) + \tilde{u}(x, y - l)\right] \leq M$$ \quad \text{if } b_0 = 0,$$

$$l^{-2}\left[\tilde{u}(x, y + l) - 2\tilde{u}(x, y) + \tilde{u}(x, y - l)\right] \leq M/l \times \text{if } b_0 > 0.$$  

Hence $u$ is a generalized solution of (1), (2), which ends the proof.

Remark 4. Using the same methods as in the proof of Theorem 1 we obtain the existence of a generalized solution of the following differential-functional system of first order partial equations

$$D_xz_i(x, y) = f_i(x, y, z(x, y), z_{(x,y)}), \ D_yz_i(x, y),$$

$$z_i(x, y) = \varphi_i(x, y), \ (x, y) \in \mathbb{R}_{\leq b_0, 0} \times R,$$

where $i = 1, \ldots, m, \ z = (z_1, \ldots, z_m), \ \varphi = (\varphi_1, \ldots, \varphi_m) : [-b_0, 0] \times R \rightarrow R^m, f = (f_1, \ldots, f_m) : [0, a_0] \times R \times R^m \times C(B, R^m) \times R \rightarrow R^m.$

References


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