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ON VARIETIES OF REGULAR *-SEMIGROUPS, II

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Following I. Chajda [1] by a *diagonal* on an algebra S we shall mean a reflexive and compatible binary relation on S . The set $\text{Ref}(S)$ of all diagonals on S forms a complete lattice with respect to set inclusion. By a *quasiorder* on S is meant a transitive diagonal on S . Analogously, the set $\text{Qua}(S)$ of all quasiorders forms a complete lattice with respect to set inclusion but $\text{Qua}(S)$ is no sublattice of $\text{Ref}(S)$ in a general case, see [2].

The aim of this paper is to describe all varieties of regular *-semigroups whose lattices of all diagonals (quasiorders) are modular, distributive and boolean.

Recall that a *regular *-semigroup* (see [3]) is an algebra $(S, \cdot, *)$ where (S, \cdot) is a semigroup and $*$ is a unary operation on S satisfying the following:

$$(1) \quad (x^*)^* = x, \quad x = xx^*x \quad \text{and} \quad (xy)^* = y^*x^*.$$

By $\mathcal{W}(i = j)$ we denote the variety of all regular *-semigroups satisfying the identity $i = j$. Terminology and notation not defined here may be found in [4] and [5].

Let S be a regular *-semigroup. For $M, N \subseteq S \times S$ we put

$$\begin{aligned} MN &= \{(ab, cd); (a, c) \in M, (b, d) \in N\}, \\ M^* &= \{(a^*, c^*); (a, c) \in M\}, \\ \bar{M} &= \{(c, a); (a, c) \in M\}. \end{aligned}$$

If $M = \{(a, c)\}$ or $N = \{(b, d)\}$, then we simply write $M = (a, c)$ or $N = (b, d)$, respectively. By a *diagonal* A on S we shall mean a reflexive regular *-subsemigroup of the direct product $S \times S$, i.e.

$$(2) \quad \text{id}_S \subseteq A, \quad AA \subseteq A \quad \text{and} \quad A^* \subseteq A.$$

By $\text{Ref}(S)$ we denote the lattice of all diagonals on S with respect to set inclusion. Denote by \vee or \wedge the join or meet in $\text{Ref}(S)$, respectively. The meet evidently coincides with the set intersection. For $M \subseteq S \times S$ we denote by $\mathbf{R}(M)$ the least diagonal on S containing M . It is easy to show the following:

$$(3) \quad (x, y) \in \mathbf{R}(M) \quad \text{if and only if} \\ x = x_1x_2 \dots x_m \quad \text{and} \quad y = y_1y_2 \dots y_m \\ \text{where either } (x_i, y_i) \in M \quad \text{or}$$

$$(x_i^*, y_i^*) \in M \text{ or } x_i = y_i \text{ for } i = 1, 2, \dots, m.$$

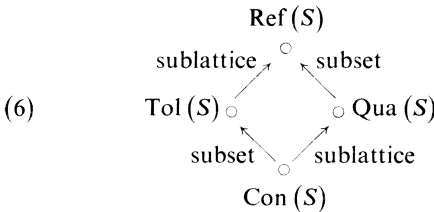
$$(4) \quad A \vee B = R(A \cup B) \text{ for } A, B \in \text{Ref}(S).$$

It is very easy to show that the mapping $A \rightarrow \bar{A}$ on $\text{Ref}(S)$ is an involution lattice automorphism on $\text{Ref}(S)$, i.e.

$$(5) \quad \bar{\bar{A}} = A, \quad \overline{A \vee B} = \bar{A} \vee \bar{B} \text{ and } \overline{A \wedge B} = \bar{A} \wedge \bar{B}.$$

for any $A, B \in \text{Ref}(S)$. Evidently $A = \bar{A}$ if and only if A is a *tolerance* (symmetric diagonal) on S . It follows from (5) that the set $\text{Tol}(S)$ of all tolerances on S is a sublattice of $\text{Ref}(S)$.

A transitive diagonal on S is said to be a *quasiorder* on S . The set $\text{Qua}(S)$ of all quasiorders on S forms a lattice with respect to set inclusion but $\text{Qua}(S)$ need not be a sublattice of $\text{Ref}(S)$. It is clear that $\text{Con}(S) = \text{Tol}(S) \cap \text{Qua}(S)$ is the lattice of all *congruences* on S . We have the following diagram:



Let us note that in [6] there are described all varieties of regular $*$ -semigroup whose tolerance (congruence) lattices are modular, distributive and boolean. In fact this paper is a continuation of [6].

Theorem 1. *The following conditions for a variety \mathcal{V} of regular $*$ -semigroups are equivalent:*

1. $\mathcal{V} \subseteq \mathcal{W}(xx^* = yy^*)$.
2. $\text{Con}(S) = \text{Ref}(S)$ for all $S \in \mathcal{V}$.
3. $\text{Con}(S)$ is a sublattice of $\text{Ref}(S)$ for all $S \in \mathcal{V}$.
4. $\text{Con}(S)$ is a sublattice of $\text{Tol}(S)$ for all $S \in \mathcal{V}$.
5. $\text{Qua}(S)$ is a sublattice of $\text{Ref}(S)$ for all $S \in \mathcal{V}$.
6. $\text{Qua}(S)$ is a sublattice of $\text{Tol}(S)$ for all $S \in \mathcal{V}$.
7. The lattice $\text{Qua}(S)$ is modular for all $S \in \mathcal{V}$.
8. The lattice $\text{Con}(S)$ is modular for all $S \in \mathcal{V}$.

Proof. $1 \Rightarrow 2$. Suppose that $S \in \mathcal{W}(xx^* = yy^*)$. We shall show that $\text{Ref}(S) \subseteq \text{Tol}(S)$. Let $A \in \text{Ref}(S)$. If $(x, y) \in A$, then by (1) and (2) we have $(y, x) = (yy^*y, xx^*x) = (xx^*y, xy^*y) \in A$. It follows from Theorem 1 of [6] that $\text{Tol}(S) = \text{Con}(S)$ and so $\text{Ref}(S) \subseteq \text{Con}(S)$. According to (6) we have $\text{Con}(S) = \text{Ref}(S)$.

$2 \Rightarrow 3, 4, 5$ and 6 . It follows from (6).

4 or 5 or $6 \Rightarrow 3$. It is clear.

- 3 \Rightarrow 1. According to (6), we obtain 3 \Rightarrow 4 and by Theorem 1 of [6] we have 4 \Rightarrow 1.
 1 \Rightarrow 7. Suppose that $S \in \mathcal{W}(xx^* = yy^*)$. It follows from 1 \Rightarrow 2 and (6) that $\text{Qua}(S) = \text{Con}(S)$ and so by Theorem 5 of [6] the lattice $\text{Qua}(S)$ is modular.
 7 \Rightarrow 8. See (6).
 8 \Rightarrow 1. Apply Theorem 5 of [6].

Theorem 2. *The following conditions for a variety \mathcal{V} of regular $*$ -semigroups are equivalent:*

1. $\mathcal{V} \subseteq \mathcal{W}(xyy^*x^* = xx^*)$.
2. The lattice $\text{Ref}(S)$ is modular for all $S \in \mathcal{V}$.
3. The lattice $\text{Tol}(S)$ is modular for all $S \in \mathcal{V}$.

Proof. 1 \Rightarrow 2. Suppose that $S \in \mathcal{W}(xyy^*x^* = xx^*)$. It is easy to show that

$$(7) \quad x e x^* = xx^*$$

for every $x \in S$ and every projection e of S (i.e. $e = e^2 = e^*$).

Let $A, B \in \text{Ref}(S)$. We shall prove that for every projection e of S we have

$$(8) \quad AB = A(e, e) B,$$

$$(9) \quad (e, e) A(e, e) = (e, e) \bar{A}(e, e),$$

$$(10) \quad (e, e) AB(e, e) = (e, e) BA(e, e),$$

$$(11) \quad ABAB \subseteq AB.$$

Identity (8). Suppose that $(a, c) \in A$ and $(b, d) \in B$. There by (1), (2) and (7) we have $(a, c)(b, d) = (a, c)(bb^*c^*c, bb^*c^*c)(e, e)(c^*c, c^*c)(b, d) \in A(e, e)B$. Consequently $AB \subseteq A(e, e)B \subseteq AB$.

Identity (9). Assume that $(a, c) \in A$. According to (1), (2) and (7), we obtain $(e, e)(a, c)(e, e) = (e, e)(ce, ce)(c^*, a^*)(ea, ea)(e, e) \in (e, e)\bar{A}(e, e)$. Thus we have $(e, e)A(e, e) \subseteq (e, e)\bar{A}(e, e)$. Analogously we can show that $(e, e)\bar{A}(e, e) \subseteq (e, e)A(e, e)$.

Identity (10). Let $(a, c) \in A$ and $(b, d) \in B$. By (1), (2), (8), (7) and (9) we have $(e, e)(a, c)(b, d)(e, e) = (e, e)(cde, cde)(d^*, b^*)(c^*, a^*)(eab, eab)(e, e) \in (e, e)\bar{B}\bar{A}(e, e) = (e, e)\bar{B}(e, e)\bar{A}(e, e) = (e, e)B(e, e)A(e, e) = (e, e)BA(e, e)$. Thus we obtain $(e, e)AB(e, e) \subseteq (e, e)BA(e, e)$ and analogously we can get $(e, e)BA(e, e) \subseteq (e, e)AB(e, e)$.

Inclusion (11). It follows from (8), (10) and (2) that $ABAB = A(e, e)BA(e, e)B \subseteq (e, e)A(e, e)AB(e, e)B \subseteq AB$.

Suppose that $A, B, C \in \text{Ref}(S)$ and $A \subseteq C$. First we shall show that

$$(12) \quad AB \cap C \subseteq A(B \cap C),$$

$$(13) \quad BA \cap C \subseteq (B \cap C)A,$$

$$(14) \quad ABA \cap C \subseteq A(B \cap C)A,$$

$$(15) \quad BAB \cap C \subseteq (B \cap C)A(B \cap C).$$

Inclusion (12). Suppose that $(x, y) \in AB \cap C$. Then by (8) and (2) we have $(x, y) = (a, c)(eb, ed)$, where $(a, c) \in A$, $(eb, ed) \in B$ and e is a projection of S . It follows from (7) and (2) that $(eb, ed) = (ea^*e, ec^*e)(x, y) \in AC \subseteq C$.

Inclusion (13). This is dual to (12).

Inclusion (14). Assume that $(x, y) \in ABA \cap C$. Then by (8) we have $(x, y) = (ue, ve)(a, c)$, where $(ue, ve) \in AB$, $(a, c) \in A$ and e is a projection of S . According to (7) and (2) we obtain $(ue, ve) = (x, y)(ea^*e, ec^*e) \in CA \subseteq C$. It follows from (12) that $(ue, ve) \in A(B \cap C)$ and so $(x, y) \in A(B \cap C)A$.

Inclusion (15). Suppose that $(x, y) \in BAB \cap C$. Then we have $(x, y) \in (b, d)AB$ where $(b, d) \in B$. Using (1), (2) and (7) we get $(xx^*, yy^*) = (bb^*, dd^*) \in B \cap C$. It follows from (1), (2), (7) and (11) that $(x, y) = (xx^*e, yy^*e)(ex, ey) \in (xx^*e, yy^*e) \cdot ABAB \subseteq (xx^*e, yy^*e)AB$, where e is a projection of S . Consequently by (8) we obtain $(x, y) = (xx^*, yy^*)(eu, ev)$, where $(eu, ev) \in AB$. According to (2), we have $(eu, ev) = (ex, ey) \in C$ and so, by (12), we get $(eu, ev) \in A(B \cap C)$. Therefore $(x, y) = (xx^*, yy^*)(eu, ev) \in (B \cap C)A(B \cap C)$.

Finally, it follows from (11), (12), (13), (14), (3) and (4) that $(A \vee B) \wedge C = (A \cup B \cup AB \cup BA \cup ABA \cup BAB) \cap C \subseteq A \cup (B \cap C) \cup A(B \cap C) \cup (B \cap C) \cdot A \cup A(B \cap C)A \cup (B \cap C)A(B \cap C) = A \vee (B \wedge C) \subseteq (A \vee B) \vee C$.

Therefore the lattice $\text{Ref}(S)$ is modular.

$2 \Rightarrow 3$. This follows from (6).

$3 \Rightarrow 1$. See Theorem 4 of [6].

Theorem 3. *The following conditions for a variety \mathcal{V} of regular *-semigroups are equivalent:*

1. $\mathcal{V} \subseteq \mathcal{W}(xyx^* = xx^*)$.
2. *The lattice $\text{Ref}(S)$ is boolean for all $S \in \mathcal{V}$.*
3. *The lattice $\text{Tol}(S)$ is boolean for all $S \in \mathcal{V}$.*
4. *The lattice $\text{Ref}(S)$ is distributive for all $S \in \mathcal{V}$.*
5. *The lattice $\text{Tol}(S)$ is distributive for all $S \in \mathcal{V}$.*

Proof. $1 \Rightarrow 2$. Suppose that $S \in \mathcal{W}(xyx^* = xx^*) \subseteq \mathcal{W}(xyy^*x^* = xx^*)$. We have

$$(16) \quad exe = e, \quad x = xex \quad \text{and} \quad xyz = xez$$

for any $x, y, z \in S$ and each projection e of S . Indeed, it follows from (7) and (1) that $xex = xex^*ex = xx^*x = x$ and $xyz = xexyzez = xez$.

Let $A, B, C \in \text{Ref}(S)$. We shall show that

$$(17) \quad ABC = AC,$$

$$(18) \quad AB \cap C = (A \cap C)(B \cap C).$$

Identity (17). According to (8), (16) we have $ABC = A(e, e)B(e, e)C = A(e, e) \cdot C = AC$.

Identity (18). Suppose that $(u, v) \in AB \cap C$. Then by (8) we obtain $(u, v) = (a, c)(e, e)(b, d)$ where $(a, c) \in A$ and $(b, d) \in B$. It follows from (16) and (2)

that $(ae, ce) = (aebe, cede) = (ue, ve) \in A \cap C$ and analogously $(eb, ed) = (eu, ev) \in B \cap C$. Hence we have $(u, v) = (ae, ce)(eb, ed) \in (A \cap C)(B \cap C)$. Therefore $AB \cap C \subseteq (A \cap C)(B \cap C) \subseteq AB \cap C$.

According to (3), (4), (17) and (18), we have $(A \vee B) \wedge C = (A \cup B \cup AB \cup BA) \cap C = (A \cap C) \cup (B \cap C) \cup (A \cap C)(B \cap C) \cup (B \cap C)(A \cap C) = (A \wedge C) \vee (B \wedge C)$.

Therefore the lattice $\text{Ref}(S)$ is distributive.

Now we shall prove that the lattice $\text{Ref}(S)$ is boolean. Let $A \in \text{Ref}(S)$. Choose a projection e of S and put $B = R((Se \times Se) \setminus A)$.

Let $u, v \in S$. According to (1) and (16), we have $(u, v) = (ue, ve)(u^*e, v^*e)^*$. It is easy to show that $(ue, ve), (u^*e, v^*e) \in A \cup B$. By (3) and (4) we have $(u, v) \in A \vee B$. Therefore $A \vee B = S \times S$.

Assume that $A \wedge B \neq \text{id}_s$. Then there exist $u, v \in S$ such that $(u, v) \in A \cap B$ and $u \neq v$. According to (3) and (16), we have $(u, v) = (a, c)(e, e)(b, d)$, where $(a, c), (b, d) \in \text{id}_s \cup ((Se \times Se) \setminus A) \cup ((Se \times Se) \setminus A)^*$. If $(a, c) \in (Se \times Se) \setminus A$, then by our assumption we obtain $(a, c) = (ae, ce) = (aeb, ced)(e, e) = (u, v)(e, e) \in A$, which is a contradiction. Thus we have $(a, c) \notin (Se \times Se) \setminus A$. If $(b, d) \in ((Se \times Se) \setminus A)^*$, then $(b^*, d^*) \in (Se \times Se) \setminus A$ and so by our assumption we have $(b^*, d^*) = (b^*e, d^*e) = (b^*ea^*, d^*ec^*)(e, e) = (u, v)^*(e, e) \in A$, a contradiction. Therefore $(b, d) \notin ((Se \times Se) \setminus A)^*$.

Consequently we have the following possibilities:

Case 1. $a = c$. Then $b \neq d$ and so $(b, d) \in (Se \times Se) \setminus A$. Hence by our assumption we have $(u, v) = (aebe, aede) = (ae, ae)$, a contradiction.

Case 2. $b = d$. Then $a \neq c$ and so $(a, c) \in ((Se \times Se) \setminus A)^* \subseteq eS \times eS$. Therefore $(u, v) = (eae, ece) = (eb, eb)$, a contradiction.

Case 3. $a \neq c$ and $b \neq d$. Then $(a, c) \in eS \times eS$ and $(b, d) \in Se \times Se$. Thus we have $u = aeb = e = ced = v$, a contradiction.

Therefore $A \wedge B = \text{id}_s$. Consequently the lattice $\text{Ref}(S)$ is boolean.

$2 \Rightarrow 4$. It is clear.

$4 \Rightarrow 5$. This follows from (6).

$5 \Rightarrow 1$. See Theorem 6 of [6].

$2 \Rightarrow 3$. Suppose that the lattice $\text{Ref}(S)$ is boolean.

According to (6), $\text{Tol}(S)$ is a sublattice of $\text{Ref}(S)$ and so the lattice $\text{Tol}(S)$ is distributive. Let $A \in \text{Tol}(S)$. Then there exists $B \in \text{Ref}(S)$ such that $A \wedge B = \text{id}_s$ and $A \vee B = S \times S$. We have $A = \bar{A}$ and so by (5) we obtain $A \wedge \bar{B} = \text{id}_s$ and $A \vee \bar{B} = S \times S$. Therefore $B = \bar{B} \in \text{Tol}(S)$. Consequently the lattice $\text{Tol}(S)$ is boolean.

$3 \Rightarrow 5$. This follows from (6).

Note. Let be a variety of regular $*$ -semigroups. If the lattice $\text{Qua}(S)$ is distributive for all $S \in \mathcal{V}$, then \mathcal{V} is trivial.

This follows from (6) and from Theorem 7 of [6].

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