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REMARK ON CAVITATION SOLUTIONS OF STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS IN ONE DIMENSION

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1. INTRODUCTION

The initial boundary-value problems for one-dimensional Navier-Stokes equations

$$(1.1)\qquad \begin{aligned}&\varrho(u_t + uu_x) + p(\varrho)_x - \mu u_{xx} = \varrho f\,,\\ &\varrho_t + (\varrho u)_x = 0\,,\quad x \in (0, 1)\,,\quad t \in (0, \infty)\end{aligned}$$

have been studied by many authors as e.g. [1]–[15]. (This list is by no means complete.) In these papers mainly existence, uniqueness and global properties of solutions to (1.1) with initial conditions

$$(1.2)\qquad u(x, 0) = u_0(x)\,,\quad \varrho(x, 0) = \varrho_0(x)\,,\quad x \in [0, 1]$$

and various types of boundary conditions as e.g. the Dirichlet boundary condition

$$(1.3)\qquad u(x, 0) = u(1, t) = 0\,,\quad t \geqq 0\,,$$

have been investigated. A common feature of all the results is that they claim or work with solutions (u, ϱ) for which

$$(1.4)\qquad \operatorname{essinf}\{\varrho(x, t);\ x \in (0, 1)\} > 0\,,\quad t \in [0, \infty)\,.$$

This restriction excludes from the consideration an important class of solutions with vacuum states, i.e. solutions, for which ϱ can be zero for x from a set of a positive measure either for fixed t or asymptotically as $t \to \infty$. Since it has been proved (see e.g. [3], [5], [10]) that for regular data (1.4) holds whenever

$$\varrho_0(x) \geqq \alpha_0 > 0\,,\quad x \in (0, 1)\,,$$

the former case can occur only if the same holds for ϱ_0, while the latter case has been indicated by the numerical experiments made by V. Casulli and A. Valli (private communication). Theoretically, in view of the results in [2], [8], [10] it is clear that if a solution of (1.1)–(1.3) with ϱ bounded from above satisfies the uniform estimate

$$(1.4)_1\qquad \operatorname{essinf}\{\varrho(x, t);\ x \in (0, 1)\} \geqq \alpha > 0\,,\quad t \in [0, \infty)$$

then for the case $f = f(x)$ the function f and the state equation $p = p(\varrho)$ must satisfy certain compatibility condition (see Proposition 2.2 below). Since for physically important constitutive relations $p = p(\varrho)$ there always exist f which do not satisfy this condition, the uniform estimate $(1.4)_1$ cannot hold in this case. On the other hand $(1.4)_1$ implies that the strong limits in $L^2(0, 1)$, $\bar{u}(x) = \lim_{t\to\infty} u(x, t)$, $\bar{\varrho}(x) = \lim_{t\to\infty} \varrho(x, t)$ satisfy stationary equations

$$\varrho\bar{u}\bar{u}_x + p(\bar{\varrho})_x - \mu\bar{u}_{xx} = \bar{\varrho}f\,,$$
$$(\bar{\varrho}\bar{u})_x = 0\,,$$
$$\bar{u}(0) = \bar{u}(1) = 0\,,$$

which clearly imply

$$\bar{u} = 0\,,$$
$$p(\bar{\varrho})_x = \bar{\varrho}f\,, \quad x \in (0, 1)\,, \quad \operatorname{essinf}\{\bar{\varrho}(x);\ x \in (0, 1)\} > 0\,.$$

We expect that at least some solutions of (1.1)–(1.3) converge to solutions of the stationary problem with cavities provided that (1.4) fails to be satisfied. This fact has motivated the present study of the cavitation solutions to the stationary (equilibrium) problem which in normalized form, can be written as

$$(1.5)\qquad p(\varrho)_x = \varrho f\,, \quad x \in (0, 1)\,,$$
$$\int_0^1 \varrho(x)\,\mathrm{d}x = 1, \quad \varrho(x) \geqq 0\,, \quad x \in (0, 1)\,,$$

where

$$(1.6)\qquad p \in C^1((0, \infty))\,, \quad p(0+) = p(0) = 0\,, \quad p'(r) > 0 \quad \text{for} \quad r > 0\,,$$
$$f = f(x)\,, \quad f \in L^\infty(0, 1)\,.$$

In Section 2 we present some examples and existence (in some sense also uniqueness) theorems for the problem (1.5).

In what follows we use the standard notation as $W^{k,p}$ or H^p for the Sobolev spaces and C^k (or C_0^k) for continuously differentiable functions (with compact support) and the like.

2. STATIONARY SOLUTIONS WITH CAVITIES

In this section we are going to investigate solutions of the problem (1.5) under the assumptions (1.6) unless specified another way. Let $P(r) = \int_1^r (p'(s)/s)\,\mathrm{d}s$, $r > 0$, $a = \int_0^1 (p'(s)/s)\,\mathrm{d}s \equiv P(0)$, $b = \int_1^\infty (p'(s)/s)\,\mathrm{d}s$, ($a = -\infty$ and/or $b = \infty$ is not excluded). Put $\Phi = P^{-1}$ for the inverse function of P and $F(x) = \int_0^x f(\xi)\,\mathrm{d}\xi$.

2.1. Definition. By a solution of the problem (1.5) we mean a function $\varrho \in C([0, 1])$ satisfying $\int_0^1 \varrho(x)\,\mathrm{d}x = 1$ and

$$\int_0^1 [p(\varrho)\,\varphi_x + \varrho f \varphi]\,\mathrm{d}x = 0 \quad \text{for any} \quad \varphi \in C_0^\infty(0, 1)\,.$$

In [2] the following result has been proved.

2.2. Proposition. *Under the assumption* (1.6) *the problem* (1.5) *has a solution* $\varrho \in W^{1,\infty}(0, 1)$ *satisfying* $\varrho(x) \geqq \alpha > 0$, *a.e.* $x \in (0, 1)$, (α = const.) *if and only if the following two conditions hold*:

$$\sup_{[0,1]} F - \inf_{[0,1]} F < b - a\,, \tag{2.1}$$

$$\int_0^1 \Phi(F(x) - \inf_{[0,1]} F + a)\,\mathrm{d}x < 1 < \int_0^1 \Phi(F(x) - \sup_{[0,1]} F + b)\,\mathrm{d}x\,. \tag{2.2}$$

2.3. Example. Let $p(\varrho) = \varrho^\gamma$, γ = const., $1 < \gamma < 2$, $f(x) \equiv f_0$ = const. > 0. Then $a = -\gamma/(\gamma - 1)$, $b = \infty$, $\Phi(\xi) = (1 + ((\gamma - 1)/\gamma)\,\xi)^{1/(\gamma-1)}$ and the conditions (2.1), (2.2) reduce to

$$f_0 < \left(\frac{\gamma}{\gamma - 1}\right)^\gamma. \tag{2.3}$$

The solution can be obtained by explicit integration, where the integration constant is chosen so that $(1.5)_2$ is satisfied. The result is

$$\varrho(x) = \left(d + \frac{\gamma - 1}{\gamma} f_0 x\right)^{1/(\gamma-1)}, \quad x \in [0, 1]\,,$$

where d is the (unique) solution of the equation

$$\left(d + \frac{\gamma - 1}{\gamma} f_0\right)^{\gamma/(\gamma-1)} - d^{\gamma/(\gamma-1)} = f_0 \frac{\gamma - 1}{\gamma}\,.$$

The situation, when (2.1), (2.2) fails to be satisfied is illustrated by the following example:

2.4. Example. The state equation and f are the same as in Example 2.3 but (2.3) is violated. More specifically, we suppose

$$f_0 > \left(\frac{\gamma}{\gamma - 1}\right)^\gamma.$$

It can be shown easily that the function ϱ defined by

$$\varrho(x) = 0 \quad \text{for} \quad 0 \leqq x \leqq 1 - \frac{\gamma}{\gamma - 1} f_0^{-1/\gamma}$$

$$\varrho(x) = \left(\frac{\gamma - 1}{\gamma} f_0\right)^{1/(\gamma-1)} \left(x + \frac{\gamma}{\gamma - 1} f_0^{-1/\gamma} - 1\right)^{1/(\gamma-1)}$$

$$\text{for} \quad 1 - \frac{\gamma}{\gamma - 1} f_0^{-1/\gamma} < x \leqq 1$$

is a solution to (1.5).

Another example shows that the solution of (1.5) might not be unique unless an additional condition is introduced.

2.5. Example. The same situation as above, but $f(x) = f_0$, $x \in [0, \frac{1}{4}) \cup [\frac{1}{2}, \frac{3}{4})$,

$f(x) = f_0$, $x \in [\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1]$. Then $F(x) = f_0 x$, $x \in [0, \frac{1}{4})$; $F(x) = f_0(\frac{1}{2} - x)$, $x \in [\frac{1}{4}, \frac{1}{2})$; $F(x) = f_0(x - \frac{1}{2})$, $x \in [\frac{1}{2}, \frac{3}{4})$; $F(x) = f_0(1 - x)$, $x \in [\frac{3}{4}, 1]$; the condition (2.2) means

$$f_0 < 4^\gamma \left(\frac{\gamma - 1}{\gamma}\right)^{1/\gamma} \equiv \varphi_0 .$$

If $f_0 > \varphi_0$ then we can construct a solution as follows. Let $x_0 \in [0, \frac{1}{4})$, $x_1 \in [\frac{1}{2}, \frac{1}{4})$. Put

$$\varrho(x) = 0 \quad \text{for} \quad x \in [0, x_0] \cup [\tfrac{1}{4} + x_0, x_1] \cup [\tfrac{3}{2} - x_1, 1] ,$$

$$\varrho(x) = \left(\frac{\gamma - 1}{\gamma}\right)^{1/(\gamma - 1)} [F(x) - F(x_0)] \quad \text{for} \quad x \in (x_0, \tfrac{1}{4} + x_0)$$

$$\varrho(x) = \left(\frac{\gamma - 1}{\gamma}\right)^{1/(\gamma - 1)} [F(x) - F(x_1)] \quad \text{for} \quad x \in (x_1, \tfrac{3}{2} - x_1) .$$

It is clear that $\varrho \in C([0, 1])$ and it satisfies $(1.5)_1$ in the sense of distributions. By computation, $\int_0^1 \varrho(x)\,dx = 1$ if and only if

$$x_0^{\gamma/(\gamma-1)} + x_1^{\gamma/(\gamma-1)} = 4^{\gamma/(1-\gamma)} + \left(\frac{3}{4}\right)^{\gamma/(\gamma-1)} - \frac{\gamma}{2(\gamma - 1)} f_0^{1/(\gamma-1)} .$$

The choice of x_0, x_1 yields an one-parameter family of solutions to (1.5). The solution can be fixed up if we prescribe

$$\int_{x_0}^{1/4 - x_0} \varrho(x)\,dx = m , \quad \int_{x_1}^{3/2 - x_1} \varrho(x)\,dx = 1 - m ,$$

where $1 - m_0 \leqq m \leqq m_0$ with $m_0 = 2^{(\gamma+1)/(\gamma-1)}((\gamma - 1)/\gamma) f_0^{1/(\gamma-1)}$. Physically this amounts to the distribution of the total mass 1 between the intervals $(x_0, \frac{1}{4} - x_0)$ and $(x_1, \frac{3}{2} - x_1)$. The mass distribution is limited by the bounds $m \leqq m_0$, $1 - m \leqq m_0$, while $m_0 > \frac{1}{2}$ since $f_0 > \varphi_0$.

The following theorem claims the existence of a solution to (1.5) with cavitation in particular case generalizing the situation in Example 2.4.

2.6. Theorem. *Let* $\int_0^1 \Phi(F(x) + a - \inf_{[0,1]} F)\,dx > 1$, $f(x) \geqq 0$ *a.e. in* $(0, 1)$ *and* $\int_0^1 f(x)\,dx < b - a$. *Then there exists a solution* $\varrho \in C([0, 1])$ *to* (1.5) *such that there exists* $x_0 \in (0, 1)$ *for which*

$$\varrho(x) = 0 , \quad x \in [0, x_0] \tag{2.4}$$
$$\varrho(x) = \Phi(F(x) - F(x_0) + a) , \quad x \in (x_0, 1] .$$

Proof. It is clear that for any fixed $x_0 \in (0, 1)$ the function ϱ defined by (2.4) satisfies $(1.5)_1$, $\varrho \in C([0, 1])$ and $\varrho(x) \geqq 0$ in $[0, 1]$. It remains to show that x_0 can be chosen so that $\int_0^1 \varrho(x)\,dx = 1$. Put $\varphi(x_0) = \int_{x_0}^1 \Phi(F(x) - F(x_0) + a)\,dx$. Since $F(0) = 0 = \inf_{[0,1]} F$, we have $\varphi(0) > 1$. As $\varphi(1) = 0$, φ is decreasing and continuous, there exists a unique $x_0 \in (0, 1)$ such that $(\int_0^1 \varrho(x)\,dx =)\ \varphi(x_0) = 1$.

A similar theorem holds for $f(x) \leqq 0$, $x \in (0, 1)$. We do not formulate it since it is quite analogous. The following theorem concerns the case when (2.2) is not satisfied in its right hand inequality.

2.7. Theorem. *Let* (2.1) *be satisfied and* $\int_0^1 \Phi(F(x) + b - \sup_{[0,1]} F)\,dx < 1$. *Then there is no solution to* (1.5).

Proof. Let there exist a solution ϱ to (1.5). Set $\Omega = \{x \in (0, 1); \varrho(x) > 0\}$. Then Ω is an opened subset of $(0, 1)$, hence $\Omega = \bigcup_{k \in J} (x_k, y_k)$ for some $x_k < y_k$, $k \in J$, where $J \subset N$. The integration of $(1.5)_1$ in (x_k, y_k) yields $\varrho(x) = \Phi(F(x) + c_k)$, $x \in (x_k, y_k)$ with some constant c_k. Since $\varrho(x_k) = 0$ (or, alternatively, $\varrho(y_k) = 0$ if $x_k = 0$ and $\varrho(0) > 0$), we have $c_k = a - F(x_k)$. Thus we find

$$\int_0^1 \varrho(x)\,dx = \sum_{x \in J} \int_{x_k}^{y_k} \Phi(F(x) - F(x_k) + a)\,dx \leqq$$

$$\leqq \sum_{k \in J} \int_{x_k}^{y_k} \Phi(F(x) + b - \sup_{[0,1]} F)\,dx \leqq \int_0^1 \Phi(F(x) + b - \sup_{[0,1]} F)\,dx < 1\,.$$

The following theorem is a uniqueness theorem which respects the specific situation illustrated by Example 2.5.

2.8. Theorem. *Let* $\varrho \in C([0, 1])$ *be a solution of* (1.5). *Write* $\Omega = (0, 1) \cap \operatorname{supp} \varrho$ *in the form* $\Omega = \bigcup_{k \in J} I_k$, *where* $I_k = (x_k, y_k)$, $k \in J$. *Then there is no other solution* $\sigma \in C([0, 1])$ *of* (1.5) *that satisfies*

$$\int_{I_k} \sigma\,dx = \int_{I_k} \varrho\,dx \quad \textit{for any} \quad k \in J\,. \tag{2.5}$$

Proof. Let $\sigma \neq \varrho$ be another solution of (1.5) satisfying (2.5). Since $\int_0^1 \sigma\,dx = \int_0^1 \varrho\,dx = 1$, we have $\sigma = 0$ a.e. in $(0, 1) \setminus \Omega$ and $\sigma \neq 0$ a.e. in I_k for any $k \in J$. Put

$$a(x) = \int_0^1 p'(\alpha\,\varrho(x) + (1 - \alpha)\,\sigma(x))\,d\alpha\,, \quad r = \varrho - \sigma\,, \quad (x \in I_k)$$

and take an arbitrary $k \in J$, but fixed. Then we have

$$(a(x)\,r)_x = rf\,, \quad x \in I_k$$

in the sense of distributions and $\int_{I_k} r\,dx = 0$. Since $\varrho > 0$, $\sigma > 0$ in I_k, we have $a > 0$ in I_k as well. Hence it is clear that

$$r(x) = \frac{a(z_k)}{a(x)} \exp\left(\int_{z_k}^{x} \frac{f}{a}\,d\xi\right) r(z_k)\,, \quad x \in I_k\,,$$

where $z_k = \frac{1}{2}(x_k + y_k)$. If $r(z_k) \neq 0$ then $\int_{I_k} r\,dx \neq 0$. Hence $r = 0$ in I_k.

Next, investigate the case when (2.2) is violated in its left hand inequality.

2.9. Theorem. *Let* $f \in C([0, 1])$, $\int_0^1 \Phi(F(x) - \inf_{[0,1]} F + a)\,dx > 1$, $\{x; F(x) = 0\} = \{0, 1\}$ *and there is a unique* $x_0 \in (0, 1)$ *such that* $f(x_0) = 0$. *Then either*

$F(x_0) > 0$ *and there exist* x_1, x_2: $0 < x_1 < x_2 < 1$ *such that the function* ϱ

defined by

$$(2.6)\qquad \varrho(x) = \begin{cases} \Phi(F(x) - F(x_1) + a), & x \in (x_1, x_2) \\ 0, & x \in [0, x_1] \cup [x_2, 1] \end{cases}$$

is a (unique) solution of (1.5), *or*

$F(x_0) < 0$ *and the exist* x_1, x_2: $0 \leqq x_1 < x_2 \leqq 1$ *such that the function*

$$(2.7)\qquad \varrho(x) = \begin{cases} \Phi(F(x) - F(x_1) + a), & x \in [0, x_1) \\ 0, & x \in [x_1, x_2] \\ \Phi(F(x) - F(x_2) + a), & x \in (x_2, 1] \end{cases}$$

is a solution of (1.5).

In the latter case, given m:

$$(2.8)\qquad 1 - \int_{x_0}^1 \Phi(F(x) - F(x_0) + a)\,\mathrm{d}x \leqq m \leqq \int_0^{x_0} \Phi(F(x) - F(x_0) + a)\,\mathrm{d}x\,,$$

there is a unique solution satisfying $\int_0^{x_1} \varrho(x)\,\mathrm{d}x = m$ *and this solution is given by* (2.7) *with suitable* x_1, x_2 *depending on* m.

Proof. 1. Let $F(x_0) > 0$. Then F is increasing in $(0, x_0)$ and decreasing in $(x_0, 1)$. Hence for any $x_1 \in (0, x_0)$ there is a unique $x_2 \in (x_0, 1)$ such that $F(x_1) = F(x_2) = \inf_{[x_1,x_2]} F$. Put $x_2 = b(x_1)$, $\varphi(x_1) = \int_{x_1}^{b(x_1)} \Phi(F(x) - F(x_1) + a)\,\mathrm{d}x$. Then $\varphi(0) = \int_0^1 \Phi(F(x) + a)\,\mathrm{d}x > 1$ since $\inf_{[0,1]} F = 0$, and $\varphi(x_0) = 0$. (b is increasing, $b(1) = 1$, $b(x_0) = x_0$). By continuity of φ there exists a (unique) $x_1 \in (0, x_0)$ such that $\varphi(x_1) = 1$. This proves the existence part of the first assertion. Let there be a continuous solution σ of (1.5) which is not identical with ϱ. Suppose that there is a $\xi \in [0, x_1] \cup [x_2, 1]$ such that $\sigma(\xi) > 0$. Then there is a maximal interval $(\alpha, \beta) \subset (0, 1)$ containing ξ and $\sigma > 0$ in (α, β). By integration and uniqueness in (α, β) we get $\sigma(x) = \Phi(F(x) - F(\alpha) + a)$, $x \in (\alpha, \beta)$, $F(\alpha) = F(\beta)$. This yields $\alpha \in [0, x_1]$, $\beta \in [x_2, 1]$. But if we had $\alpha < x_1$ or $\beta > x_2$ it would be $\int_\alpha^\beta \sigma(x)\,\mathrm{d}x > 1$. Thus $\sigma = 0$ in $[0, x_1] \cup [x_2, 1]$. Now, if $\Omega = \bigcup_{x \in J} (\alpha_k, \beta_k) = \{x \in (x_1, x_2); \sigma(x) > 0\}$ with (α_k, β_k) being connected components of Ω, then $F(\alpha_k) = F(\beta_k)$, from where $x_1 \leqq \alpha_k < x_0 < \beta_k \leqq x_2$. Hence $\Omega = (\alpha, \beta)$, where $\alpha = \inf_{k \in J} \alpha_k$, $\beta = \sup_{k \in J} \beta_k$. If $\alpha > x_1$ then clearly $\beta < x_2$ and $\int_\alpha^\beta \sigma(x)\,\mathrm{d}x < 1$. We find $\alpha = x_1$, $\beta = x_2$, from where $\sigma = \varrho$ in (x_1, x_2).

2. Let $F(x_0) < 0$. It is clear that for any $x_1 \in [0, x_0]$, $x_2 \in [x_0, 1]$ we have $F(x_1) = \inf_{[0,x_1]} F$, $F(x_2) = \inf_{[x_2,1]} F$. Then the function (2.7) satisfies (1.5) possibly except for $\int_0^1 \varrho(x)\,\mathrm{d}x = 1$. Putting

$$\varphi(x_1, x_2) = \int_0^{x_1} \Phi(F(x) - F(x_1) + a)\,\mathrm{d}x + \int_{x_2}^1 \Phi(F(x) - F(x_2) + a)\,\mathrm{d}x,$$

we find

$$\varphi(x_0, x_0) = \int_0^{x_0} \Phi(F(x) - F(x_0) + a)\,\mathrm{d}x + \int_{x_0}^1 \Phi(F(x) - F(x_0) + a)\,\mathrm{d}x = \int_0^1 \Phi(F(x) - \inf_{[0,1]} F + a)\,\mathrm{d}x > 1$$

since $F(x_0) = \inf_{[0,1]} F$, and $\varphi(0, 1) = 0$. Besides φ is the increasing function of x_1 and decreasing in x_2. Hence, given m satisfying (2.8) there is a unique pair $(x_1, x_2) \in \in [0, x_0] \times [x_0, 1]$ such that

$$\int_0^{x_1} \varrho(x)\, dx = m\,, \quad \int_{x_2}^1 \varrho(x)\, dx = 1 - m\,.$$

To prove uniqueness, suppose that there is a $\xi \in (x_1, x_2]$, $\sigma(\xi) > 0$. If (α, β) is the connected component of $\Omega = \{x;\ \sigma(x) > 0\}$ containing ξ then either $\sigma(\alpha) = 0$ or $\sigma(\beta) = 0$. Let e.g. $\sigma(\alpha) = 0$. Then $\sigma(x) = \Phi(F(x) - F(\alpha) + a)$, $x \in (\alpha, \beta)$. If $\alpha < x_0$ then $F(x) < F(\alpha)$ for $x \in (\alpha, x_0)$ and σ is no more defined. Hence $x_0 \leqq \alpha < \beta \leqq 1$. If $\beta < 1$ then $F(\alpha) = F(\beta)$, which implies $\alpha = \beta$. So we have $\beta = 1$. On the other hand, if $\sigma(\beta) = 0$ then similarly we get $\alpha = 0$, $\beta < x_0$. Hence Ω consists of two components $(0, \beta)$ and $(\alpha, 1)$, where $\beta < x_0$, $\alpha > x_0$. Now it is clear that if, for a given m satisfying (2.8), we require $\int_\alpha^1 \sigma(x)\, dx = 1 - m$, then $\beta = x_1$, $\alpha = x_2$ and $\sigma = \varrho$ in $[0, 1]$.

The preceding theorem requires very special f but it illustrates the way the solutions are constructed. Now we state the general existence theorem.

2.10. Theorem. *Suppose that* (2.1) *holds. Then for any* $f \in L^1(0, 1)$ *satisfying* $\int_0^1 \Phi(F(x) - \sup_{[0,1]} F + b)\, dx > 1$ *there exists a solution* $\varrho \in C([0, 1])$ of (1.5).

Proof. First suppose that there are $f_n \in L^1(0, 1)$ such that $f_n \to f$ in $L^1(0, 1)$ and for any n there exists a solution ϱ_n of (1.5) with $f = f_n$. Let n be arbitrary but fixed and $\Omega = \{x \in (0, 1);\ \varrho_n(x) > 0\}$. Then $\Omega = \bigcup_{j \in J} (\alpha_j, \beta_j)$ for some $0 \leqq \alpha_j < \beta_j \leqq 1$, with J countable. If $\varrho_n(x) > 0$ in $[0, 1]$ then by Proposition 2.2 $\varrho_n(x) = \Phi(F(x) + c_n)$ $x \in [0, 1]$ with a constant c_n such that $\int_0^1 \Phi(F(x) + c_n)\, dx = 1$. If we had $c_n \to \infty$ then this would not be possible. Hence $|\varrho_n(x)| \leqq \text{const} < \infty$ independently of x and n. Let on the other hand there is a $\xi \in [0, 1]$ such that $\varrho(\xi) = 0$. Take an arbitrary interval (α_j, β_j). Then either $\varrho(\alpha_j) = 0$ or $\varrho(\beta_j) = 0$. Suppose e.g. $\varrho(\alpha_j) = 0$. Then we have

$$\varrho_n(x) = \Phi(F(x) - F(\alpha_j) + a) \leqq \Phi(\sup_{[0,1]} F - \inf_{[0,1]} F + a) < \infty\,,$$
$$x \in (\alpha_j, \beta_j)\,.$$

This proves $\sup_n |\varrho|_{L^\infty(0,1)} < \infty$. Further, we have $p(\varrho_n)_x = f_n$. So $p(\varrho_n)$ is bounded in $W^{1,1}(0, 1)$ and hence compact in $C([0, 1])$. Choose a subsequence, denoted again by ϱ_n such that $p(\varrho_n)$ converges uniformly in $[0, 1]$ to some $q \in C([0, 1])$. Then $\varrho_n \to p^{-1}(q)$ pointwise in $[0, 1]$; from Lebesgue dominated theorem $\varrho_n \to \varrho \equiv \equiv p^{-1}(q)$ in $L^2(0, 1)$. Clearly, ϱ is a solution of (1.5). Now let $f \in L^1(0, 1)$ be arbitrary. Let f_n be polynomials in $[0, 1]$ such that $f_n \to f$ in $L^1(0, 1)$. So it suffices to show that (1.5) has a solution if f is a polynomial. Let f be a polynomial. For $k < \sup_{[0,1]} F$ put $\Omega_k = \{x \in (0, 1);\ F(x) > k\}$. It is clear that for $k_1 > k_2$ we have $\Omega_{k_1} \subset \Omega_{k_2}$. Besides, taking the decomposition of Ω_k with connected components $(\alpha_{kj}, \beta_{kj})$,

$0 \leqq \alpha_{kj} < \beta_{kj} \leqq 1$, $j \in J_k$, we have J_k countable, $\Omega_k = \bigcup_{j \in J_k} (\alpha_{kj}, \beta_{kj})$ and $F(\alpha_{kj}) = F(\beta_{kj}) = k$ except for, possibly, when $\alpha_{kj} = 0$ or $\beta_{kj} = 1$ for some j. Define

$$\varrho_k(x) = \Phi(F(x) - k + a)\,, \quad x \in (\alpha_{kj}, \beta_{kj})\,, \quad j \in J_k\,, \tag{2.9}$$

$\varrho_k(x) = 0$ elsewhere in $(0, 1)$ with the only exception when $\alpha_{kj} = 0$ or $\beta_{kj} = 1$ for some j and $F(0) \neq k$ or $F(1) \neq k$, respectively. In this case we define ϱ_k by continuity from the adjacent interval $(\alpha_{kj}, \beta_{kj})$.

It is clear that the function ϱ_k defined by (2.9) satisfies $(1.5)_1$, and $\varrho_k(x) \geqq 0$ in $[0, 1]$ for any $k < \sup_{[0,1]} F$. In the case $\int_0^1 \Phi(F(x) - \inf_{[0,1]} F + a)\,\mathrm{d}x < 1$ we have the existence theorem (see Proposition 2.2). Hence, suppose $\int_0^1 \Phi(F(x) - \inf_{[0,1]} F + a)\,\mathrm{d}x \geqq 1$. It is clear that

$$\bigcap_{n=1}^{\infty} \Omega_{k-(1/n)} = \Omega_k \cup \{x \in (0, 1);\ F(x) = k\}\,.$$

Since the measure of a level set of a polynomial, unless it is constant (but this case is solved by Theorem 2.6), is zero we have $\operatorname{meas} \Omega_k = \lim_{n\to\infty} \operatorname{meas} \Omega_{k-(1/n)}$ which yields "continuity" of the function $k \to \operatorname{meas} \Omega_k$. Define $I(k) = \int_{\Omega_k} \Phi(F(x) - k + a)\,\mathrm{d}x$. Then $I(k)$ is a continuous function defined for $k < \sup_{[0,1]} F \equiv s$ and

$$\lim_{k \uparrow s} I(k) = 0\,, \quad I(\inf_{[0,1]} F) = \int_0^1 \Phi(F(x) - \inf F + a)\,\mathrm{d}x \geqq 1\,.$$

Hence there exists a $k \in [\inf_{[0,1]} F, \sup_{[0,1]} F)$ such that the function (2.9) satisfies (1.5).

All of the preceding results are derived under the assumption $\sup_{[0,1]} F - \inf_{[0,1]} F \leqq b - a$. This assumption is not empty if and only if

$$\int_0^{\infty} \frac{p'(s)}{s}\,\mathrm{d}s < \infty\,. \tag{2.10}$$

Now we are going to remove this restriction.

2.11. Theorem. *Let* $f \in L^1(0, 1)$, $k_0 = \sup_{[0,1]} F + a - b$ *and let* (2.10) *hold.*

Then a solution of (1.5) *exists if*

$$\int_{F(x) > k_0} \Phi(F(x) - k_0 + a)\,\mathrm{d}x \geqq 1\,. \tag{2.11}$$

Proof. Let (2.11) be satisfied. Define $\Omega_k = \{x \in (0, 1);\ F(x) > k\}$,

$$\varrho_k(x) = \begin{cases} \Phi(F(x) - k + a) & \text{for } x \in \Omega_k \\ 0 & \text{for } x \in (0, 1) \setminus \Omega_k \end{cases} \tag{2.12}$$

$$\varrho_k(0) = \varrho_k(0+)\,, \quad \varrho_k(1) = \varrho_k(1-)\,, \quad \text{where} \quad k < \sup_{[0,1]} F\,.$$

Write Ω_k in the form $\Omega_k = \bigcup_{j \in J} (\alpha_{kj}, \beta_{kj})$. Then $F(\alpha_{kj}) = k$ if $\alpha_{kj} \neq 0$ and $F(\beta_{kj}) = k$ if $\beta_{kj} \neq 1$. It is clear that the function (2.12) satisfies $(1.5)_1$ and $\varrho_k(x) \geqq 0$, $x \in [0, 1]$.

Define

$$(2.13) \qquad \varphi(k) = \int_{\Omega_k} \Phi(F(x) - k + a)\,dx \quad \text{for} \quad k \in [k_0, k_1],$$

where $k_1 = \sup\limits_{[0,1]} F$ and investigate φ on $[k_0, k_1]$. Note that it may happen $\varphi(k_0) = \infty$. From (2.13) it is immediate that φ is decreasing and $\varphi(k_1) = 0$. Clearly, $\varrho_k(x) \geqq \varrho_l(x)$ for $k \leqq l$ and all x, and $\varphi(k) = \int_0^1 \varrho_k(x)\,dx$. By classical properties of Lebesque integral we find $\lim\limits_{k \to l} \varphi(k) = \varphi(l)$ for any $k, l \in [k_0, k_1]$. Thus φ is continuous, $\varphi(k_1) = 0$, $\varphi(k_0) = \int_{F(x) > k_0} \Phi(F(x) - k_0 + a)\,dx \geqq 1$. Hence there exists (a unique) k^* such that $\varphi(k^*) = 1$. This means that $\varrho_{k^*}(x)$ is a solution to (1.5).

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