Manoj Changat

On order and geodesic alignment of a connected bigraph

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 4, 713–715

Persistent URL: http://dml.cz/dmlcz/102501

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

ON ORDER AND GEODESIC ALIGNMENT OF A CONNECTED BIGRAPH

MANOJ CHANGAT, Cochin

(Received August 13, 1990)

In this paper, it is shown that the geodesic alignment on the vertex set V of a finite connected bipartite graph G is the join of order alignments with respect to all possible canonical orderings on V.

1. INTRODUCTION

An alignment on a set X is a family $\mathscr{L}$ of distinguished subsets of X, called *convex sets*, satisfying the following axioms.

$A_1(a)$: $\emptyset$ is convex.

$A_1(b)$: X is convex.

A_2: The arbitrary intersection of convex sets is convex.

A_3: The union of any family of convex sets, totally ordered by inclusion is again convex. $(X, \mathscr{L})$ is called *an aligned space*. Note that the axiom A_3 is trivially satisfied, if X is finite. For any subset S of X, the smallest convex set containing S is called the *convex hull of* S, denoted by $\mathscr{L}(S)$.

If X is a partially ordered set (poset) $(P, \leqq)$, $A \subseteq P$ is said to be *order convex*, if for any pair of points $a, b \in A$, the order interval $[a, b] = \{z \in P \mid a \leqq z \leqq b$ or $b \leqq z \leqq a\}$ is contained in A. The collection of all order convex sets of P form the order alignment on P.

If X is the vertex set V of a finite connected graph, there is the geodesic alignment on V, where a subset K of V is said to be *geodesically convex* or d-convex, if for every pair of vertices $x, y \in K$, the interval $I(x, y) \subseteq K$, where

$$I(x, y) = \{z \mid z \text{ lies in a shortest } x - y \text{ path in } G\}$$
$$= \{z \mid d(x, z) + d(z, y) = d(x, y)\}, \text{ and } d \text{ is the natural metric of the graph.}$$

If $(\mathscr{L}_i)_{i \in I}$ is a collection of alignments on X, then the smallest alignment R on X, containing all $\mathscr{L}_i$'s is called the *join of* $\mathscr{L}_i$*'s in the lattice of all alignments on* X. It is shown that $R = \bigcap_{i \in I} \mathscr{L}_i(A)$, for all finite subsets A of X. If this holds for all subsets of X, then $R = \bigvee_{i \in I} \mathscr{L}_i$ is called the strong join of $\mathscr{L}_i$'s. If X is finite then $R = \bigvee_{i \in I} \mathscr{L}_i$ is trivially the strong join of $\mathscr{L}_i$'s. See [1], for actual developments on alignments.

We call a poset $(P, \leqq)$ a *graded poset*, if there is a height function $h: P \to \mathbb{Z}$, such that

H_1: If $u \leqq v$, then $h(u) \leqq h(v)$.

H_2: If v covers u then $h(v) = h(u) + 1$.

In this paper we consider the order alignment and geodesic alignment on a finite connected bipartite graph G.

2. CANONICAL ORDERING ON THE VERTEX SET V OF G

With respect to any vertex u of G, we can order the vertex set V as follows.

For $i = 0, 1, \ldots, d(G) - 1$, we direct the edges between $N_i(u)$ and $N_{i+1}(u)$ from $N_{i+1}(u)$ to $N_i(u)$, where $d(G)$ is the diameter of G and $N_i(u)$ is the i^{th} level of u in G, namely $N_i(u) = \{v \in V \mid d(u, v) = i\}$. Defining $v \leqq_u w$, whenever there exists a directed path from w to v, gives a poset $(V, \leqq_u)$. This poset is graded with the height function $h_u(v) = d(u, v)$ for $v \in V$, i.e., $h_u(v) = i$, for any vertex v in $N_i(u)$. Since G is connected, we have $u \leqq_u v$, for all $v \in V$, and so u is the universal lower bound of the poset $(V, \leqq_u)$. This kind of ordering on the vertex set of a finite connected bigraph has been considered by Mulder [2] known as canonical ordering of G with respect to the vertex u. The set of all canonical orderings of G is denoted by $C(G)$.

Theorem 2.1. (Mulder [2]) *A graph G is connected and bipartite if and only if G is the digraph of a finite graded poset with universal lower bound.*

Let E denote any canonical ordering of G, and D_E denote the corresponding order alignment on V. Let $\mathscr{L}$ denote the geodesic alignment on the vertex set V of G. Now we have the main theorem.

Theorem 2.2. *The geodesic alignment $\mathscr{L}$ on the vertex set V of a finite connected bipartite graph G is the join of order alignments D_E, with respect to all canonical orderings E on V. That is, $\mathscr{L} = \bigvee_{E \in C(G)} D_E$.*

Proof. Suppose $K \in \mathscr{L}$. Now every geodesically convex (d-convex) subset of V induces a connected subgraph of G. Therefore the subgraph induced by K of G is connected and bipartite, since G is bipartite. Therefore by Theorem 2.1, K is a graded poset with a universal lower bound u. Now let E denote the canonical ordering on G with u as the universal lower bound, and K be a subposet of $(V, \leqq_u)$. Clearly $K \in D_E$. Therefore $\mathscr{L} \subseteq \bigvee_{E \in C(G)} D_E$. Conversely let $K \in \bigvee_{E \in C(G)} D_E$. Let $K = \{u_1, u_2, \ldots, u_n\} \subseteq V$. Let E_i denote the canonical ordering on G with u_i as universal lower bound, for $i = 1, \ldots, n$. Therefore $K \in D_{E_i}$ for every $i = 1, \ldots, n$.

For any pair

$$u_i, u_j \in K , \quad \text{if} \quad u \in I(u_i, u_j) ,$$

then

$$d(u, u_i) \leqq d(u_i, u_j)$$

i.e.,

$$u \leqq_{u_i} u_j .$$

That is $u_i \leqq u \leqq u_j \Rightarrow u \in [u_i, u_j] \subseteq K$, since $K \in D_{E_i}$ i.e., $I(u_i, u_j) \subseteq K$, for every $u_i, u_j \in K$, which shows that K is d-convex and hence $\bigvee_{E \in C(G)} D_E \subseteq \mathscr{L}$, which completes the proof of the theorem.

Acknowledgement. The author wishes to thank Professor T. Thrivikraman, for his advice, during the preparation of this paper.

References

[1] *R. Jamison-Waldner:* A perspective on abstract convexity classifying alignments by varieties in "Convexity and Related Combinatorial Geometry" (D. Kay and M. Breen, Eds.) pp. 113—150, Decker, New York, 1982.

[2] *H. M. Mulder:* The Interval Function of a Graph. Dissertation, Math. Centre Tracts 132, Amsterdam, 1980.

Author's address: Department of Mathematics and Statistics, Cochin University of Science and Technology, Cochin 682 022, India.