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SOME REMARKS ON NUMERICAL SOLUTION OF LINEAR PROBLEMS

MIROSLAV FIEDLER

In my talk, I would like to present some observations about three topics from linear numerical analysis.

The first observation concerns the inversion of large symmetric matrices having a significant number of zero elements. Let us have a symmetric positive definite matrix A. Assume that for some suitable ordering of its rows and columns and for a suitable decomposition into blocks A can be expressed in the partitioned form $A = (A_{ij})$, $i, j = 1, \ldots, r$, where the graph corresponding to non-zero blocks $A_{ij} \neq 0$ (with r vertices $1, 2, \ldots, r$ and exactly those nondirected edges ij for which $A_{ij} \neq 0$) is without cycles, i.e. each its component is a tree.

It is then possible to show that (under some assumptions fulfilled for positive definite matrices) the inverse matrix $A^{-1} = B = (B_{ij})$ can easily be found completely if we know the diagonal blocks B_{ii} and the blocks B_{ij}, $i \neq j$, corresponding to the non-zero blocks of A.

Two finite algorithms can be found for obtaining the mentioned characterizing blocks B_{ii} and B_{ij}.

One of them is the algorithm corresponding to the block elimination with suitably chosen ordering of eliminated block rows and columns. The second is described in [2].

The advantage of these special methods is in saving the storage capacity considerably, similarly as in the case of inverting a tridiagonal matrix by well known methods.

As the second topic I would like to give a convergence theorem for the generalized Jahn-Collar perturbation method [5] for solving the eigenvalue problem for symmetric matrices.

This method can be described as follows:

Let A be a symmetric matrix, D the diagonal of A, B the off-diagonal part of A, so that

$$A = D + B .$$

If all diagonal elements of D are different from each other, then there exists a skew-symmetric matrix S such that

$$SD - DS = \tfrac{1}{2}B .$$

This matrix $S = (s_{ij})$ is easily found as $s_{ij} = (\frac{1}{2}b_{ij})/(d_j - d_i)$, $i \neq j$.

Now, the matrix $U = (E + S)(E - S)^{-1}$ exists and is unitary. Let us denote by $\varphi(A)$ the matrix UAU^*, by $N(C)$ the Schur norm $(\sum_{i,j} |c_{ij}|^2)^{\frac{1}{2}}$ if $C = (c_{ij})$.

By analogous methods as in [3] the following theorem can be proved:

Theorem. *Let*

$$c(A) = \min_{i,j,i \neq j} |d_i - d_j| .$$

If

$$\frac{N(B)}{c(A)} < k \doteq 0{,}4 ,$$

then the sequence $A_0 = A$, $A_{k+1} = \varphi(A_k)$ *exists, converges quadratically and* $\lim_{k \to \infty} A_k = A_\infty$ *is diagonal.*

The same estimate holds in the case that D is the block-diagonal part of A and $c(A)$ is the minimal distance of proper values of two different diagonal blocks of D.

An analogous method can be used for solving the generalized eigenvalue problem $\det (P - \lambda Q) = 0$ by iteration if both P and Q are symmetric, and such that their off-diagonal parts (or the block off-diagonal parts) are relatively small.

In this case we transform $P - \lambda Q$ into another symmetric matrix $(E + Y) . (P - \lambda Q)(E + Y')$ where $Y = (y_{ik})$ is given (if possible) by $y_{ii} = 0$, $y_{ik} = (p_{ii}q_{ik} - p_{ik}q_{ii}) . (p_{kk}q_{ii} - p_{ii}q_{kk})^{-1}$ for $i \neq k$, $i, k = 1, \ldots, n$.

The third topic is the problem of estimating proper values of matrices and of separation of one simple proper value. As is well known, the Gershgorin theorem asserts that all proper values of a matrix $A = (a_{ij})$ are contained in the union of disks $\bigcup_i K_i$ where

$$K_i = \{z \mid |z - a_{ii}| \leqq \sum_{j \neq i} |a_{ij}|\} .$$

Since the proper values of A remain unchanged under diagonal similarity, this theorem applied to the matrix $B = (d_i a_{ij} d_j^{-1})$, $d_i > 0$, yields that all proper values of A are contained in the union $\bigcup_i K'_i$ with

$$K'_i = \left\{z \mid |z - a_{ii}| \leqq \sum_{j \neq i} |a_{ij}| \frac{d_i}{d_j}\right\}$$

as well.

To present another interpretation, let us introduce the notion of the *spectral set* of a square n-rowed matrix $C = (c_{ij})$ as the set of all ordered n-tuples of diagonal elements of such diagonal matrices D for which $\det (C - D) = 0$. If C is nonnegative and D is nonnegative, we shall speak about the *non-negative spectral set* of C.

Now, the validity of the following equivalent estimate can be easily shown:

Let $\varrho_1, \ldots, \varrho_n$ be an element of the nonnegative spectral set of the matrix

$$B = \begin{pmatrix} 0, & |a_{12}|, & \ldots, & |a_{1n}| \\ |a_{21}|, & 0, & \ldots, & |a_{2n}| \\ \ldots & \ldots & \ldots & \ldots \\ |a_{n1}|, & |a_{n2}|, & \ldots, & 0 \end{pmatrix}.$$

Then all proper values of the matrix $A = (a_{ij})$ are contained in the union $\bigcup_i K_i^{(\varrho)}$ where $K_i^{(\varrho)} = \{z \mid |z - a_{ii}| \leqq \varrho_i\}$.

More precisely, the set of ordered radii of the Gershgorin disks of all matrices diagonally similar to A is (in the case that A is irreducible) equal to the nonnegative spectral set of B.

Since spectral sets of both the matrix and its transpose are equal, the last formulation is symmetric with respect to transposition while the original Gershgorin theorem is not.

In most cases, it is more important to know an estimate of the distance of a diagonal element of A, say a_{11}, from the nearest proper value of A. If there exists a generalized Gershgorin disk $K_1^{(\varrho)}$ which is disjoint with all remaining disks $K_i^{(\varrho)}$, $i > 1$, then R. S. Varga [6] and J. Todd showed how to obtain the smallest isolated Gershgorin disk with the center a_{11}.

In [4], stronger but numerically more complicated estimates and separation theorems were proved: If the original matrix A is expressed in the partitioned form

$$A = \begin{pmatrix} a_{11} & a_1 \\ a_2' & A_{22} \end{pmatrix}$$

then these estimates are in some sense invariant under the more general block-diagonal similarity

$$\begin{pmatrix} d_1 & 0 \\ 0 & D_2 \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & a_1 \\ a_2' & A_{22} \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

To formulate these estimates, we denote by $g(x)$ any fixed norm in the $(n - 1)$-space of complex row-vectors, by $g'(y')$ the adjoint norm of column-vectors and by $g(C)$ the corresponding operator norm of the matrix C.

The mentioned estimate shows that, under the assumption that $B = A_{22} - a_{11}E_2$ with the identity matrix E_2 is nonsingular and

$$\sigma = (|1 + a_1 B^{-2} a_2'| - |a_1 B^{-1} a_2'|\, g(B^{-1}))^2 - \\ - 4g(a_1 B^{-1})\, g'(B^{-1} a_2')\, |a_1 B^{-1} a_2'|\, g(B^{-1}) > 0\,,$$

then there is exactly one simple proper value of A in the disk

$$|z - a_{11}| \leqq \frac{2|a_1 B^{-1} a_2'|}{|1 + a_1 B^{-2} a_2'| + |a_1 B^{-1} a_2'|\, g(B^{-1}) + \sqrt{\sigma}}\,.$$

It is possible to show that the known Laguerre estimate [1] (without the assumption of separation and unicity) for the existence of a root of an algebraic equation yields the estimate (tr C being the trace of C)

$$|z - a_{11}| \leqq \frac{n|a_1 B^{-1} a_2'|}{|1 + a_1 B^{-2} a_2' - (a_1 B^{-1} a_2') \operatorname{tr} B^{-1}|}$$

which is asymptotically n-times less accurate than the preceding one.

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