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# THE $L_2$-NORM IN THE STUDY OF ERROR PROPAGATION IN INITIAL VALUE PROBLEMS

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(to topic b)

We consider[1]) computations in $r + 1$ dimensional rectangular grids where one coordinate — called time $t$ — has been distinguished by the posing of the initial value problem. The grid points are $P_\nu^n := (nh; \nu_1 h_1, \ldots, \nu_r h_r)$ and the values of a vector-valued function $u$ on the grid are correspondingly denoted by $u_\nu^n$. The grid parameters $h_\varrho$ in the spacial directions $x_\varrho$, $\varrho = 1(1)\,r$, are given functions of the time step $h$ and tend to zero with $h$.

We assume that by the nature of the problem we may restrict our considerations in each grid level $t = nh$ to $r$-dimensional grid domains $L_h$ with $N_h$ grid points, $N_h$ finite for $h > 0$. The values of a grid function $u$ for $t = nh$ are measured by a norm $\|u\|_h^n$. The two ordinarily used norms are (see e.g. [1]):

1) The maximum norm: $\quad {}^{\infty}\|u\|_h^n := \max_{\nu \in L_h} \|u_\nu^n\|$.

2) The (discretized) $L_2$-norm: $\quad {}^{2}\|u\|_h^n := \sqrt{\dfrac{1}{N_h} \sum_{\nu \in L_h} \|u_\nu^n\|^2}$.

The right-hand norm $\|\ldots\|$ is some vector norm for the function vectors $u_\nu^n$, its choice is of no influence on our considerations.

If an initial value problem for a partial differential equation is solved numerically by a $m + 1$ level discretization method in a rectangular grid, the error vectors $E_\nu^n$ obey a partial difference equation which is of $m$-th order with respect to $t$:

$$\sum_\tau A_\tau^0 E_{\nu+\tau}^n = \sum_\tau A_\tau^1 E_{\nu+\tau}^{n-1} + \ldots + \sum_\tau A_\tau^m E_{\nu+\tau}^{n-m} + \varepsilon_\nu^n\,, \tag{1}$$

$$\nu \in L_h\,, \quad n = m, m + 1, \ldots$$

The $\varepsilon_\nu^n$ are the local errors, both from discretization and round-off. The coefficient

[1]) Comp. [1] for more details of the problem and notation.

matrices $A_\tau^\mu$ will in general depend on the grid parameters $h$ and $h_\varrho$ and on the independent variables $t$ and $x_\varrho$. The summations over $\tau$ may be over fixed vicinities of 0 or over the whole grid domain $L_h$.

It is assumed that the initial value problem for the partial difference equation (1) is properly posed, i.e. that (1) may be solved for $E^n$ at each point of the grid, $n \geqq m$. As an immediate consequence the stability properties of (1) with respect to the inhomogeneities $\varepsilon_\nu^n$ are equivalent to those with respect to initial values. Therefore the $m+1$ level algorithm is stable in a norm $\|\ldots\|_h^n$ if the solutions of the homogeneous equation (1) admit an estimate

$$\text{(2)} \qquad \|E\|_h^n \leqq S \max_{\mu=1(1)m} \|E\|_h^{l-\mu}, \quad m \leqq l \leqq n,$$

$$\text{for} \quad m \leqq n \leqq T/h, \quad T > 0 \quad \text{fixed}, \quad 0 < h \leqq h_0 .$$

The important aspect of (2) is, of course, that the estimate must be uniform in $h$ as $h$ approaches zero and the number $N_h$ of grid points in the domain $L_h$ tends to infinity.

As a consequence of (2) the accumulated error $E^n$ of the original algorithm may be estimated by

$$\text{(3)} \qquad \|E\|_h^n \leqq K\left[\sum_{\mu=0}^{m-1} \|e\|_h^\mu + \sum_{l=m}^{n} (\|d\|_h^l + \|r\|_h^l)\right]$$

where $e_\nu^\mu$ are the starting errors, $d_\nu^n$ the local discretization errors and $r_\nu^n$ the local round-off errors of the computation.

When $\|\ldots\|_h^n$ in (2) has been the $L_2$-norm the estimate (3) is also in this norm. This implies a bound on the individual errors $E_\nu^n$ for $t = nh$ which is $\sqrt{N_h}$ times as large as the one obtained from (2) and (3) in the max-norm. ($N_h \to \infty$ as $h \to 0$!) Nevertheless, the large majority of stability investigations for partial difference equations of type (1) have been based on the $L_2$-norm for two reasons:

a) The stability analysis in the $L_2$-norm is usually much easier than in the max-norm.

b) The error growth found in practical computations with $L_2$-stable algorithms never exceeded that which was to be expected for max-stability even if the particular algorithms were not max-stable at all (like the Lax-Wendroff scheme).

We will shortly analyze the reasons for this phenomenon $b$.

We will separate the treatment of discretization and round-off errors because they are of a different structure (although they both propagate according to (1)): The local discretization errors $d_\nu^n$ can ordinarily be regarded as discretizations of a smooth function $d(t, x)$ while the local round-off errors $r_\nu^n$ are ordinarily realizations of a random variable.

Let us first look at the global discretization error $D_\nu^n$: It has been shown in [2] that for a $p$-th order method $D_\nu^n$ possesses an asymptotic expansion

$$\text{(4)} \qquad D_\nu^n(h) = h^p D_0(t_n, x_\nu) + h^{p+1} D_1(t_n, x_\nu) + \ldots +$$
$$+ h^P D_{P-p}(t_n, x_\nu) + \hat{D}_\nu^n \quad \text{with} \quad {}^2\|\hat{D}\|_h^n = O(h^{P+1})$$

if the original problem as well as the algorithm are sufficiently differentiable[2]) and if the algorithm is $L_2$-stable. The functions $D_l(t, x)$ *do not depend upon h*, they are bounded in the regions considered.

Therefore, if $\sqrt{N_h} = O(h^{-q})$, $q > 0$, we have from (4)

$$(5) \qquad \max_{n \leqq T/h, \nu \in L_h} \|D_\nu^n(h)\| = O(h^P) \quad \text{if} \quad P \geqq p + q - 1$$

since ${}^\infty\|\hat{D}\|_h^n \leqq \sqrt{N_h} \cdot {}^2\|\hat{D}\|_h^n$ (see [1], Theorem 4.4). As $P$ depends only on the differentiability properties of the problem[2]), for sufficiently smooth problems the growth of the discretization error in $L_2$-stable algorithms does not differ from that in max-stable ones[3]).

With respect to the local round-off errors $r_\nu^n$ we assume that they are independent random variables with mean zero. It is then reasonable to obtain a bound for the covariance matrix of the accumulated round-off error $R_\nu^n$ instead of a bound for $R_\nu^n$ itself since the first one will much better indicate the size of the error which is likely to occur (comp. e.g. [4]).

As a solution of (1), $R_\nu^n$ depends linearly on the local errors $r_\nu^n$:

$$R_\nu^n = \sum_{l=m}^{n} \sum_{\lambda \in L_h} G_{\nu,\lambda}^{n,l} r_\lambda^l \,.$$

This implies (because of the independence of the various $r_\nu^n$)

$$\operatorname{covar}(R_\nu^n) = \sum_{l=m}^{n} \sum_{\lambda \in L_h} G_{\nu,\lambda}^{n,l} \operatorname{covar}(r_\lambda^l) \, (G_{\nu,\lambda}^{n,l})^T$$

or

$$(6) \qquad \|\operatorname{covar}(R_\nu^n)\| < n\sigma^2 \max_{l} \sum_{\lambda \in L_h} \|G_{\nu,\lambda}^{n,l}\|^2$$

where $\sigma^2$ is a common bound for the covariance matrices of the $r_\nu^n$. But the $L_2$-stability of (1) is equivalent to the uniform boundedness of $\sum_\lambda \|G_{\nu,\lambda}^{n,l}\|^2$ for arbitrary $l \leqq n$ and $\nu \in L_h$ as $h \to 0$. Hence (6) implies for an $L_2$-stable algorithm

$$(7) \qquad \max_{n \leqq T/h, \nu \in L_h} \|\operatorname{covar}(R_\nu^n)\| \leqq M \frac{\sigma^2}{h} \,.$$

Thus the bound for a deviation which is not exceeded with given probability grows only like $1/\sqrt{h}$.

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[2]) For the concise differentiability assumptions see [2].

[3]) The above reasoning was employed — in a somewhat different and more special form — by STRANG ([3]).

As (5) and (7) are identical with the estimates which could have been obtained immediately for max-stable algorithms we have shown that *under the assumptions stated* $L_2$-stability guarantees the same restricted growth of the error as max-stability. Only in extreme situations a $L_2$-stable scheme which is not max-stable will behave worse than a max-stable one.

### *References*


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