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ON THE ESTIMATE OF THE ERROR OF QUADRATURE
FORMULAE

MILOŠ ZLÁMAL

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Let $f(x)$ be a function defined on the interval $\langle 0, 1 \rangle$ and let

$$(1) \quad I(f) = \sum_{k=1}^m c_k f(x_k) \quad (0 \leq x_1 < x_2 < \dots < x_m \leq 1)$$

be any quadrature formula which is exact for every polynomial of degree less or equal n , $n \leq 2m - 1$.¹⁾ Our aim is to investigate the error

$$(2) \quad E(f) = \int_0^1 f(x) dx - I(f)$$

in the class $W_2^{(r)}$ with

$$(3) \quad r \leq n + 1.$$

By $W_2^{(r)}$ we denote the class of functions $f(x)$ such that $f^{(r-1)}(x)$ is absolutely continuous on the interval $\langle 0, 1 \rangle$ and $f^{(r)}(x) \in L_2(0, 1)$. We will find an explicit expression for

$$(4) \quad E^* = \sup_{\substack{f \in W_2^{(r)} \\ \|f^{(r)}\|_{L_2} \leq 1}} |E(f)|$$

so that for the error $E(f)$ it holds the following exact estimate

$$(5) \quad |E(f)| \leq E^* \|f^{(r)}\|_{L_2}.$$

Let $P_{r-1}(x)$ be a polynomial of degree $r - 1$ such that

$$\begin{aligned} P_{r-1}(0) &= f(0), \quad P'_{r-1}(1) = f'(1), \quad P''_{r-1}(0) = f''(0), \dots, \\ P_{r-1}^{(r-1)}(0) &= f^{(r-1)}(0) \text{ if } r \text{ is odd,} \\ P_{r-1}^{(r-1)}(1) &= f^{(r-1)}(1) \text{ if } r \text{ is even.} \end{aligned}$$

¹⁾ The necessary and sufficient condition is $\sum_{k=1}^m c_k x_k^i = (1/i + 1)$, $i = 0, 1, \dots, n$.

As $r - 1 \leq n$ we have $E(P_{r-1}) = 0$ and if we set $f(x) - P_{r-1}(x) = \varphi(x)$ it holds $E(f) = E(\varphi)$. The function $\varphi(x)$ satisfies the boundary conditions

$$(6) \quad 0 = \varphi(0), \quad 0 = \varphi'(1), \quad 0 = \varphi''(0), \quad \dots, \quad 0 = \begin{cases} \varphi^{(r-1)}(0) & \text{if } r \text{ is odd,} \\ \varphi^{(r-1)}(1) & \text{if } r \text{ is even.} \end{cases}$$

Now let us denote by ${}^0W_2^{(r)}$ the space of functions from $W_2^{(r)}$ which satisfy the boundary conditions (6). The expression

$$(\varphi, \psi) = \int_0^1 \varphi^{(r)}(x) \cdot \psi^{(r)}(x) \, dx$$

is a scalar product and ${}^0W_2^{(r)}$ with the norm

$$\|\varphi\| = (\varphi, \varphi)^{1/2} = \left\{ \int_0^1 [\varphi^{(r)}(x)]^2 \, dx \right\}^{1/2}$$

is a separable Hilbert space. The functional $E(\varphi)$ is linear in ${}^0W_2^{(r)}$ so that we have $|E(\varphi)| \leq E \cdot \|\varphi\|$ where E is the norm of the functional $E(\varphi)$. As $E(\varphi) = E(f)$ and $\|\varphi\| = \|f^{(r)}\|_{L_2}$ we have $|E(f)| \leq E \|f^{(r)}\|_{L_2}$ and obviously

$$E^* = E.$$

To find E let us consider the sequence

$$\varphi_j(x) = \sqrt{2} \left(\frac{2}{\pi}\right)^r \frac{1}{(2j-1)^r} \sin(2j-1) \frac{\pi}{2} x, \quad j = 1, 2, \dots$$

This sequence is complete and orthonormal in ${}^0W_2^{(r)}$. For this reason it holds the well known formula

$$E^{*2} = E^2 = \sum_{j=1}^{\infty} [E(\varphi_j)]^2.$$

Now it is sufficient to compute $E(\varphi_j)$ and to sum. We have

$$E(\varphi_j) = \sqrt{2} \left(\frac{2}{\pi}\right)^r \frac{1}{(2j-1)^{2r}} \left[\frac{2}{\pi} \frac{1}{2j-1} - \sum_{k=1}^m c_k \sin(2j-1) \pi \frac{x_k}{2} \right]$$

and

$$\begin{aligned} (E^*)^2 &= 2 \left(\frac{2}{\pi}\right)^{2r} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^{2r}} \left[\frac{2}{\pi} \frac{1}{2j-1} - \sum_{k=1}^m c_k \sin(2j-1) \pi \frac{x_k}{2} \right]^2 = \\ &= 2 \left(\frac{2}{\pi}\right)^{2r} \left\{ \frac{4}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^{2r+2}} - \frac{4}{\pi} \sum_{k=1}^m c_k \sum_{j=1}^{\infty} \frac{\sin(2j-1) \pi x_k/2}{(2j-1)^{2r+1}} + \right. \\ &\quad \left. + \sum_{k,l=1}^m c_k c_l \sum_{j=1}^{\infty} \frac{1}{2} \frac{1}{(2j-1)^{2r}} [\cos(2j-1) \pi \frac{1}{2}(x_k - x_l) - \right. \\ &\quad \left. \left. - \cos(2j-1) \pi \frac{1}{2}(x_k + x_l)] \right\}. \end{aligned}$$

If we use Euler polynomials $E_r(x)$ (see [1], pp. 23–29 and 66) the final result can be written in this way:

$$(7) \quad (E^*)^2 = (-1)^r \frac{2^{2r-2}}{(2r-1)!} \left\{ \sum_{k,l=1}^m c_k c_l [E_{2r-1}(\frac{1}{2}|x_k - x_l|) - E_{2r-1}(\frac{1}{2}(x_k + x_l))] - \frac{4}{r} \sum_{k=1}^m c_k E_{2r}(\frac{1}{2}x_k) - \frac{4}{r(2r+1)} E_{2r+1}(0) \right\}.$$

We do not apply (7) to known quadrature formulae to get numerical results as the reader can find these values in [2] where they were computed in a special way. Special cases of the formula (7) ($r = 1, 2, 3$) are proved in [3].

Bibliography

- [1] *N. E. Nörlund*: Vorlesungen über Differenzenrechnung. Berlin 1944.
 [2] *C. M. Никольский*: Квадратурные формулы (Quadrature Formulae). Москва 1958.
 [3] *В. Ф. Кузютин*: Об оценке ошибки квадратурной формулы (On the Estimate of the Error of Quadrature Formulae). Методы вычислений, выпуск II, Ленинград 1963.

Souhrn

О ОДHAДУ ЧYBY KBAДPAТУРНÝХ ФОРМУЛÝ

MILOŠ ZLÁMAL

Nechť $f(x)$ je funkce definovaná na intervalu $\langle 0, 1 \rangle$ a

$$I(f) = \sum_{k=1}^m c_k f(x_k), \quad 0 \leq x_1 < x_2 < \dots < x_m \leq 1$$

libovolná kvadrurní formule, která je přesná pro každý polynom stupně menšího nebo rovného n , $n \leq 2m - 1$. Cílem článku je vyšetřit chybu

$$E(f) = \int_0^1 f(x) dx - I(f)$$

ve třídě funkcí $W_2^{(r)}$ ($r \leq n + 1$). Pod $W_2^{(r)}$ rozumíme třídu funkcí $f(x)$ takových, že $f^{(r-1)}(x)$ je absolutně spojitá na intervalu $\langle 0, 1 \rangle$ a $f^{(r)}(x) \in L_2(0, 1)$. Je nalezen explicitní výraz (7) pro

$$E^* = \sup_{\substack{f \in W_2^{(r)} \\ \|f^{(r)}\|_{L_2} \leq 1}} |E(f)|,$$

takže pro chybu $E(f)$ platí přesný odhad

$$|E(f)| \leq E^* \cdot \|f^{(r)}\|_{L_2}.$$

Резюме

ОБ ОЦЕНКЕ ОШИБКИ КВАДРАТУРНЫХ ФОРМУЛ

МИЛОШ ЗЛАМАЛ (MILOŠ ZLÁMAL)

Пусть $f(x)$ — функция, определенная на промежутке $\langle 0, 1 \rangle$ и

$$I(f) = \sum_{k=1}^m c_k f(x_k), \quad 0 \leq x_1 < x_2 < \dots < x_m \leq 1$$

произвольная квадратурная формула, которая является точной для каждого многочлена степени меньшей или равной n , $n \leq 2m - 1$. Цель работы — исследовать ошибку

$$E(f) = \int_0^1 f(x) dx - I(f)$$

в классе функций $W_2^{(r)}$ ($r \leq n + 1$). Под $W_2^{(r)}$ мы подразумеваем класс функций $f(x)$ таких, что $f^{(r-1)}(x)$ абсолютно непрерывна в промежутке $\langle 0, 1 \rangle$ и $f^{(r)}(x) \in L_2(0, 1)$. Найдено явное выражение (7) для

$$E^* = \sup_{\substack{f \in W_2^{(r)} \\ \|f^{(r)}\|_{L_2} \leq 1}} |E(f)|,$$

так что для ошибки $E(f)$ имеет место точная оценка

$$|E(f)| \leq E^* \|f^{(r)}\|_{L_2}.$$

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