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Aplikace matematiky, Vol. 13 (1968), No. 2, 174–176

Persistent URL: http://dml.cz/dmlcz/103151

AN EXTENSION OF MILLER'S ALGORITHM

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A powerful computational algorithm for evaluating the most rapidly decreasing solution of a second-order homogeneous linear difference equation was published in 1952 by J. C. P. MILLER ([1], page xvii) in connection with the tabulation of modified Bessel functions. Since then, various writers have applied the algorithm to other special functions. An error analysis was supplied by the present writer in [2], and quite recently GAUTSCHI [3] has examined the relation of the algorithm to classical results in the theory of continued fractions.

The present investigation stems from the observation that Miller's algorithm can be regarded as a procedure for solving a tridiagonal set of simultaneous linear algebraic equations. From this more general standpoint the algorithm can be recast into a new form which enables the correct number of recurrence steps to be determined automatically without appeal to an asymptotic or other analytical formula. In this respect it resembles an algorithm proposed recently by SHINTANI [4].

The new formulation has the further advantages of (i) being applicable to inhomogeneous difference equations, (ii) lending itself readily to powerful error analyses. There seems to be no alternative method of comparable power available at present for computing solutions of inhomogeneous equations in the case when forward recurrence and backward recurrence are both unstable.

Let the given difference equation be denoted by

$$a_r y_{r-1} - b_r y_r + c_r y_{r+1} = d_r \,, \tag{1}$$

where a_r, b_r, c_r, and d_r are given functions of the integer variable r. We assume that the general solution has the form

$$y_r = Af_r + Bg_r + h_r \,, \tag{2}$$

in which A and B are arbitrary constants, and the complementary functions f_r, g_r, and the particular solution h_r have the properties $f_0 \neq 0$, $g_r \neq 0$ for all sufficiently large r, and

$$f_r/g_r \to 0 \,, \quad h_r/g_r \to 0 \,, \quad (r \to \infty) \,. \tag{3}$$

We seek the solution y_r having the property

(4) $$y_r/g_r \to 0\,, \quad (r \to \infty)\,,$$

and satisfying the *normalizing condition* $y_0 = k$ for an arbitrarily assigned value of the constant k [1]).

It is well known that direct use of (1) as a recurrence relation for generating $y_2, y_3, \ldots$ from values of y_0 and y_1 (if available) is an unstable procedure. Essentially, each computational rounding error introduces into the numerical solution a small multiple of f_r and a small multiple of g_r, and in consequence of (4) the latter ultimately grows faster than the wanted solution.

It may also happen in the inhomogeneous case that f_r grows more rapidly than y_r in the direction of decreasing r. In this event recurrence by use of (1) is unstable in this direction too.

Analogous work in the numerical solution of linear differential equations suggests that a stable way of solving the present problem is to treat it directly as a boundary-value problem. We are already given the value of y_0. We assume that $y_N = 0$ for some large integer N. Then the system of $N + 1$ equations

(5) $$a_r y_{r-1}^{(N)} - b_r y_r^{(N)} + c_r y_{r+1}^{(N)} = d_r\,, \quad (r = 1, 2, \ldots, N - 1)\,,$$
$$y_0^{(N)} = k\,, \quad y_N^{(N)} = 0\,,$$

generally determines the $N + 1$ unknowns $y_0^{(N)}, y_1^{(N)}, \ldots, y_N^{(N)}$.

Convergence theorem. *Provided that equations* (5) *are non-singular,* $y_r^{(N)} \to y_r$ *as* $N \to \infty$, *r being fixed* [5].

The most convenient way of solving equations (5) is by simple forward elimination and back-substitution. The process can be expressed by

(6) $$p_{r+1} y_r^{(N)} - p_r y_{r+1}^{(N)} = e_r\,, \quad (r = 1, 2, \ldots, N - 1)\,,$$

where p_r is the solution of the homogeneous form of (1) with the conditions $p_0 = 0$ and $p_1 = 1$, and e_r is given by $e_0 = k$ and

(7) $$c_r e_r = a_r e_{r-1} - d_r p_r\,, \quad (r > 0)\,.$$

The minimum value of N needed to achieve specified accuracy in the wanted solution y_r can be determined automatically during the elimination by use of the

[1]) The method can be extended to allow for a more general normalizing condition of the form

$$\sum_{r=0}^{\infty} m_r y_r = 1\,,$$

in which the m_r are given numbers.

formulas [5]

$$y_r - y_r^{(N)} = E_N p_r\,, \quad E_N = \sum_{s=N}^{\infty} \frac{e_s}{p_s p_{s+1}}\,, \tag{8}$$

the last series necessarily being convergent. Generally the convergence is rapid, so that $y_r - y_r^{(N)} \doteqdot e_N p_r/(p_N p_{N+1})$. Accordingly, the forward recurrence of the p's is terminated as soon as N is large enough to ensure that the last quantity falls below a specified tolerance for a given range of values of r. The back-substitution is then carried out by use of (6), beginning with $y_N^{(N)} = 0$.

It can be shown that the algorithm is quite stable, except when f_0 is unduly small. In this case the problem is ill-posed and another normalizing condition should be used in place of $y_0 = k$.

Full details of the method are given in [5], together with applications to Bessel functions, Anger-Weber functions, Struve functions, and the solution of ordinary differential equations in Chebyshev series by Clenshaw's method [6]. Extensions to difference equations of higher orders are under investigation.

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