Aplikace matematiky

Olof B. Widlund

Error estimates for partial differential equations

Aplikace matematiky, Vol. 13 (1968), No. 2, 208--210

Persistent URL: http://dml.cz/dmlcz/103159

ERROR ESTIMATES FOR PARTIAL DIFFERENTIAL EQUATIONS

O. B. WIDLUND

How to derive error estimates for solutions of stable difference approximations to initial value problems for linear partial differential equations with smooth initial values has been well known for many years. Cf. RICHTMYER [12]. Thus Lax's equivalence theorem tells us that, for any initial value in some appropiiate space, the solution of a stable difference equation will, when the mesh size goes to zero, converge to the solution of the corresponding continuous problem. Lax's theorem does not give any information about the rate of convergence. However, it is quite easy to give realistic error bounds provided certain natural extra conditions are fulfilled and the initial value function is sufficiently smooth. Using results due to STRANG [13] we can extend this technique to quasi-linear equations with sufficiently smooth solutions.

Then the question as to what can be said when the initial values are not so smooth or even discontinuous arises. Some older results, dealing with special difference schemes for parabolic problems, are reported in JUNCOSA & YOUNG [5], [6], [7] and WASOW [14]. Very few results seem to be published as yet on the very important problem of estimating the error when difference methods are applied to linear or non-linear hyperbolic equations with initial values with jump discontinuities. In the linear case we will then have contact discontinuities travelling along the characteristics and in the non-linear case there will be shocks. These discontinuity lines can be considered as internal boundaries. It is therefore not surprising that these problems have much in common with certain mixed initial boundary value problems. Earlier results in dealing with this kind of mixed problem are reported in PARTER [10].

It is the purpose of this note to announce progress in this field of research by mathematicians working in Sweden during the last academic year. Very few details will be given in this paper. We believe that reprints will be available at the time of the Liblice meeting.

Of the papers we are announcing, PEETRE & THOMÉE [11] deals with the most general class of initial value problems for linear equations. The authors introduce the concept of strongly correctly posed systems of partial differential equations. This class includes all well posed problems which are of parabolic type, first order or have constant coefficients. The basic technical tools come from the theory of interpolation spaces. There are two main results in this paper. The first gives an error estimate, in terms of the accuracy of the scheme and the smoothness of the initial value function, for explicit difference approximations to general systems of strongly correctly posed problems. The second results deals with parabolic systems. It is shown that one can improve the estimates for the convergence rates by taking into account the well known fact that solutions of parabolic equations are very smooth.

HEDSTROM [3] uses a very refined analysis of Fourier transforms to investigate the convergence rates of scalar, constant coefficient difference schemes in one space dimension. His estimates, which are shown to be the best possible, are phrased in terms of the accuracy of the scheme and the continuity properties of the initial values which are supposed to belong to certain Lipschitz classes. The main emphasis is on difference schemes which approximate first order hyperbolic equations and are stable in L^2 but not in the max.-norm. For these schemes Hedstrom derives precise estimates of the maximum norm of the error. The author also studies the error when the initial values have jump discontinuities, deriving constant coefficient counterparts of some results due to ANDERSSON [1]. Cf. the discussion of Andersson's work below. A third part contains an improvement of the results of Peetre & Thomée in the case of scalar, constant coefficient, parabolic equations. In that case the difference schemes are supposed to fulfill the FRITZ JOHN [4] condition.

WIDLUND [17] is an extension of Hedstrom's results in the parabolic case to general systems which are parabolic in the sense of Petrowskiĭ. The difference schemes are supposed to be parabolic, a concept generalizing the Fritz John condition which was introduced in Widlund [15]. The results are improvements of those of Peetre & Thomée in the parabolic case. The proofs rely heavily on earlier results [16] on the properties of the fundamental solution of parabolic difference schemes. There is also a discussion on how one often, in practical cases, can manipulate with the initial values so that one can take full advantage of the accuracy of the difference schemes.

Finally we announce three papers on hyperbolic equations. The results are all based on clever transformations of the equations and on simple L^2-estimates.

Andersson [1] deals with the question of estimating the error for difference approximations to a linear hyperbolic equation of first order with variable coefficients, the solution of which has a contact discontinuity. The author shows that there is a very great difference between dissipative and non-dissipative stable difference schemes. (For a definition and interesting results on such schemes we refer to KREISS [8].) When using a difference scheme the sharp discontinuity of the solution of a continuous problem will, of course, be spread out. Andersson is able to show that this phenomena only affect an interval of length $O(h|\log h|)$ around the discontinuity if the difference scheme is dissipative. The error can be split up into two components, one of which corresponds to the error in the case of smooth initial data, while the other has the property of dying off very quickly when we move away from the characteristic through the initial discontinuity. A non-dissipative scheme shows a much worse behaviour. A series of numerical experiments support the conjecture that these results cannot be improved considerably. The investigation provides us with an accurate tool for a comparison between different schemes.

KREISS & LUNDQVIST [9] study the size of the error introduced by incorrect extra boundary data for a difference approximation to a hyperbolic equation defined for

$\{x \geqq 0\} \times \{t \geqq 0\}$. Suppose that the characteristic of the differential equation is directed towards the left. Then no boundary condition should be introduced for $x = 0$. But when we want to use a difference scheme one usually has to introduce extra boundary conditions for $x = 0$ and also often for $x = -h, -2h$, etc. (h is the mesh size.) The main result of this paper tells us that the error can be split up into two components just as in Andersson's paper. The first of these is of the same size as for an ordinary initial value problem, the other one will die out when we get away from the boundary, if the difference scheme is contractive. The first component will, in fact, dominate when $x > O(h|\log h|)$. A difference scheme is contractive if its characteristic polynomial (symbol) fulfills a certain condition. All dissipative schemes, consistent to a hyperbolic equation, are contractive. ELVIUS & KREISS [2] study difference schemes for non-linear hyperbolic equations. The continuous problem is supposed to have the form of a single conservation law. If the difference scheme is conservative, accurate of the order $2s + 1$ and dissipative of the order $2s + 2$ then an initial value which consists of two constant levels with a jump between them will be propagated with the correct shock speed. The shock will be smeared out only over an interval of length $O(h|\log h|)$. Outside of this interval the solution will be accurate to any given accuracy. These results can be extended to any initial value which has a jump discontinuity. A long series of experiments supports the theoretical studies.

References

[1] *Andersson, M. Y. T.*, "On difference schemes for hyperbolic equations with discontinuous initial values", preprint, Uppsala 1967.

[2] *Elvius, T. & Kreiss H. O.*, to appear, Uppsala 1967.

[3] *Hedstrom, G.*, to appear, Göteborg 1967.

[4] *John, F.*, Comm. Pure Appl. Math. *5* (1952), 155—211.

[5] *Juncosa, M. L. & Young, D.*, J. Soc. Indust. Appl. Math. *1* (1953), 111—135.

[6] *Juncosa, M. L. & Young, D.*, Proc. Amer. Math. Soc. *5* (1954), 168—174.

[7] *Juncosa, M. L. & Young, D.*, Proc. Cambridge Philos. Soc. *53* (1957), 448—461.

[8] *Kreiss, H. O.*, Comm. Pure Appl. Math. *17* (1964), 335—353.

[9] *Kreiss, H. O. & Lundquist, E.*, "On difference approximations with wrong boundary values", preprint, Uppsala 1967.

[10] *Parter, S.*, Numer. Math. *4* (1962), 277—292.

[11] *Peetre, J. & Thomée, V.*, "On the rate of convergence for discrete initial-value problems", preprint, Göteborg 1967.

[12] *Richtmyer, R. D.*, "Difference methods for initial-value problems", Interscience Tracts in Pure and Applied Mathematics, Tract *4*, New York, 1957.

[13] *Strang, G.*, Numer. Math. *6* (1964), 37—46.

[14] *Wasow, W.*, Math. Tables, Aids Comput. *12* (1958), 43—55.

[15] *Widlund, O. B.*, Math. Comp. *19* (1965), 1—13.

[16] *Widlund, O. B.*, Numer. Math. *8* (1966), 186—202.

[17] *Widlund, O. B.*, "On the rate of convergence for parabolic difference schemes", to appear, Uppsala 1967.

O. B. Widlund, Department of Computer Sciences, Sturegatan 4B, Uppsala, Sweden.