

Jozef Zelenka

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ALGORITHMY

18. sfbesj

ALGORITHMUS FOR EVALUATION OF SPHERICAL BESSEL FUNCTIONS

JOZEF ZELENKA, Ústav kovových materiálov SAV, Bratislava

Presented procedure, *sfbesj*, is an algorithmus for evaluation of the spherical Bessel functions, defined as

$$(1) \quad j_l(z) = \sqrt{\left(\frac{\pi}{2z}\right)} J_{l+1/2}(z)$$

where  $l$  is a positive integer or zero and  $J_\nu(z)$  denotes the (“cylindrical”) Bessel function (of the 1st kind) of the order of  $\nu$ . It is to note that on the basis of eq. (1), the *sfbesj* procedure can be used for calculation of the Bessel function of the half-integer order, too.

The evaluation of our function is based on well-known relations<sup>1)</sup> namely

$$(2) \quad j_{i-1}(z) + j_{i+1}(z) = \frac{2i+1}{z} j_i(z), \quad i > 0$$

(recurrence is started with  $j_0(z) = \sin z/z$ ,  $j_1(z) = \sin z/z^2 - \cos z/z$ ) and

$$(3) \quad J_\nu(z) = \sum_{i=0}^{\infty} (-1)^i \frac{(z/2)^{\nu+2i}}{i! \Gamma(\nu+i+1)}$$

The last one is valid for the Bessel functions in general. Taking into account eqs. (1) and (3) as well as the relations  $\Gamma(x+1) = x \Gamma(x)$ ,  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$  ( $\Gamma(x)$  is the Euler gamma function) and setting  $\nu$  for  $l + \frac{1}{2}$  we obtain

$$(4) \quad j_l(z) = \frac{(z/2)^l}{\frac{3}{2} \cdot \frac{5}{2} \dots \nu} \sum_{i=0}^{\infty} (-1)^i \frac{(z/2)^{2i}}{i! (\nu+1)(\nu+2)\dots(\nu+i)}$$

<sup>1)</sup> See any standard text on the theory of Bessel equation, e.g. G. N. Watson, “Theory of Bessel Functions”, 2nd edition, Camb. Univ. Press, New York 1945.

For the reason of obtaining high accuracy for any relation of the argument and the order of the  $j_l(z)$  it seems to be convenient to calculate it according to the recurrence formula (2) for  $l \leq z$  and by summing of the series (4) for  $l > z$ . In the alternating series two successive terms are taken together in each turn, the summation is continued until the current value of the added term is less than  $10^{-7}$  times the value of the sum.

**real procedure** *sfbesj*(*l, z*); **value** *l, z*; **integer** *l*; **real** *z*;

**comment** *sfbesj* evaluates the spherical Bessel function  $j_l(z) = \sqrt{(\pi/2z)} \cdot J_{l+1/2}(z)$  ( $l \geq 0, z \geq 0$ ) by using of the recurrence formula

$$j_{i-1}(z) + j_{i+1}(z) = \frac{2i+1}{z} j_i(z), \quad i > 0$$

$$\left( j_0(z) = \frac{\sin z}{z}, j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \right)$$

for  $l \leq z$ , if  $l > z$  the truncated summation of the series

$$j_l(z) = \sqrt{\left(\frac{\pi}{2z}\right)} \sum_{i=0}^{\infty} (-1)^i \frac{(z/2)^{l+1/2+2i}}{i! \Gamma(l + \frac{1}{2} + i + 1)}$$

is employed;

**begin real** *p, koef, sum*; **integer** *i*;

**if**  $l \leq \text{entier}(z)$  **then**

**begin**

**if**  $z \leq 10^{-8}$  **then** *sum* := 1 **else begin** *sum* :=  $\sin(z)/z$ ; *p* :=  $\cos(z)/z$  **end**;

*l* :=  $l + l$ ;

**for** *i* := 2 **step** 2 **until** *l* **do**

**begin** *koef* := *sum*; *sum* := *koef* ×  $(i-1)/z - p$ ; *p* := *koef* **end**

**end**

**else**

**begin**

*p* :=  $l + .5$ ; *z* :=  $z/2$ ; *koef* := 1;

**for** *i* := 1 **step** 1 **until** *l* **do** *koef* := *koef* ×  $z/(i + .5)$ ;

*z* :=  $z \times z$ ; *sum* := *koef* ×  $(1 - z/(p + 1))$ ;

**for** *i* := 2, *i* + 2 **while** *koef* >  $\text{abs}(\text{sum}) \times 10^{-7}$  **do**

**begin** *p* :=  $p + 2$ ; *koef* := *koef* ×  $z \uparrow 2 / ((i-1)/i / (p-1) / p)$ ;

*sum* := *sum* + *koef* ×  $(1 - z/(i+1)) / (p+1)$

**end**

**end**;

*sfbesj* := *sum*

**end of** *sfbesj*;

The procedure was tested by GIER computer (30 bits for mantissa, 10 bits for exponent, 0.12 msec for floating point addition) for  $l \leq 24$ ,  $z \leq 24$ . In this range average execution time  $\sim 25$  msec

accuracy 8 significant digits usually (7 at least)

| typical results | calculated             | tabulated <sup>2)</sup> |
|-----------------|------------------------|-------------------------|
| $j_4(3)$        | $5.61497144_{10} - 2$  | $5.614971433_{10} - 2$  |
| $j_4(20)$       | $5.04761492_{10} - 2$  | $5.047615_{10} - 2$     |
| $j_{20}(3)$     | $2.39422493_{10} - 16$ | $2.394224927_{10} - 16$ |
| $j_{20}(20)$    | $3.83248497_{10} - 2$  | $3.832485_{10} - 2$     |

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<sup>2)</sup> Tables of Spherical Bessel Functions, vol. I and II, Camb. Univ. Press, New York 1947.