REMARKS ON ANDĚL'S PAPER "ON MULTIPLE NORMAL PROBABILITIES OF RECTANGLES"

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This paper contains some brief remarks on the preceding Anděl's paper [1]. First, we present here some simplifications in the bound Z_t for the remainder of the series suggested in [1]. Second, we show here an interesting special case, in which the series in question has only non-negative terms.

The contents of the present paper is very simple; however, for better orientation, its main points are arranged into separate Assertions.

This paper is an appendix to [1], so that we shall use its system of notation, only with some complements. Unexplained symbols, as well as the motivation and the basic results, may be found in [1].

Thus we deal with a regular covariance matrix $\mathbf{G} = \|g_{ij}\|_{i,j=1}^{n}$; in addition to [1], denote its characteristic roots, arranged in an increasing order, by $\gamma_1 \leqq \gamma_2 \leqq \ldots \ldots \leqq \gamma_n$. Further, the inverse of $\mathbf{G}$ is $\mathbf{G}^{-1} = \mathbf{Q} = \|q_{ij}\|_{i,j=1}^{n}$, and the characteristic roots of $\mathbf{G}$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$; for the sake of simplicity, and in addition to [1], suppose also $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$.

1. SIMPLIFICATIONS IN THE BOUND Z_t FOR THE REMAINDER

The bound Z_t, given by formula (7) in [1], for the remainder of the series $\sum_{k=0}^{\infty} c_k$ contains a quantity m defined by

$$m = \min(\lambda_1, \lambda_2, \ldots, \lambda_n, \; q_{11}, q_{22}, \ldots, q_{nn}) \, .$$

However, there is a simpler expression for m.

Assertion 1. $m = \min(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1$ *(in the present notation).*

Proof. By a well known theorem on the roots of a quadratic form (see e.g. [6], Chapter X, § 7) we have

$$\lambda_1 = \min_{x \neq 0} \frac{x'Qx}{x'x} \leqq \frac{x'Qx}{x'x}$$

for each vector $x \neq 0$. Choosing now the vector x whose i-th coordinate is 1, the remaining coordinates being 0, we get $\lambda_1 \leqq q_{ii}$ for each $i = 1, \ldots, n$, which proves the assertion.

However, the computation of characteristic roots of a matrix uses to be rather tedious. Therefore, we may use a simplification given by the following

Assertion 2. *If we are willing to accept a larger bound for the remainder in place of Z_t, we may replace m in the formula for Z_t by any number λ for which $\lambda \leqq \lambda_1$. Or, we may put $\lambda = 1/\gamma$ where γ is any number for which $\gamma \geqq \gamma_n$.*

Proof. The first part is obvious, the second part follows simply by noting that $\gamma \geqq \gamma_n$ implies $1/\gamma \leqq 1/\gamma_n = \lambda_1$.

The advantage of the modification in Assertion 2 lies in that it is much easier to find some bounds λ or γ for characteristic roots than to find the roots themselves, and there is a number of very simple methods for doing it (see e.g. Parodi's book [8]). The simplest instance is expressed in the following

Assertion 3. *We may put* $\gamma = \max_{1 \leqq i \leqq n} \sum_{j=1}^{n} |g_{ij}|$.

Proof. Follows immediately from a well known theorem on the location of characteristic roots in the so called Geršgorin circles (see e.g. [8], Chapter III, § 1, Corollary 2).

2. A SPECIAL CASE WITH A NON-DECREASING SERIES

Throughout this section we shall suppose that the matrix $G = \|g_{ij}\|_{i,j=1}^{n}$ has the following special form: $g_{11} = g_{22} = \ldots = g_{nn} = 1$; $g_{ij} = g_i g_j$ for $i \neq j$; $i, j = 1, \ldots, n$, where the numbers g_i satisfy $0 \leqq g_i < 1$ for $i = 1, \ldots, n$.

Covariance matrices of this type occur frequently in statistical publications, see e.g. [2], [3], [4], [7], [9], and others. In particular, note that covariance matrices of positively equicorrelated random variables belong to this type.

The inverse of a matrix of this type may be found quite easily.

Assertion 4. *If G has the above mentioned form, then*

$$Q = G^{-1} = D^{-1} - \frac{D^{-1}gg'D^{-1}}{1 + g'D^{-1}g}, \tag{1}$$

where $\mathbf{D}$ *is the diagonal matrix with the numbers* $1 - g_1^2, 1 - g_2^2, \ldots, 1 - g_n^2$ *in its diagonal, and* $\mathbf{g}$ *is the column vector with coordinates* $g_1, \ldots, g_n$. *Explicitly*:

$$(2) \qquad q_{ii} = \frac{1}{1 - g_i^2} - \frac{g_i^2}{(1 - g_i^2)^2 \left[1 + \sum_{k=1}^{n} (1 - g_k^2)^{-1} g_k^2\right]} \quad \text{for} \quad i = 1, \ldots, n\,,$$

$$(3) \qquad q_{ij} = - \frac{g_i g_j}{(1 - g_i^2)(1 - g_j^2)\left[1 + \sum_{k=1}^{n} (1 - g_k^2)^{-1} g_k^2\right]} \quad \text{for} \quad i \neq j\,;$$

$$i, j = 1, \ldots, n\,.$$

Proof. It suffices to note that $\mathbf{G} = \mathbf{D} + \mathbf{gg}'$, and to check directly that the matrix given by (1) multiplied by $\mathbf{G} = \mathbf{D} + \mathbf{gg}'$ gives $\mathbf{I}$ (the unit matrix).

Assertion 5. *If* $\mathbf{G}$ *has the above mentioned form, then in the series* $P(A) = \sum_{k=0}^{\infty} c_k$ *(given by* (4), (5) *in Anděl* [1]) *all terms* c_k *are non-negative. Hence, for each* $t = 0, 1, 2, \ldots, \sum_{k=0}^{t} c_k \leqq P(A)$.

Proof. By formula (5) in [1], c_k is equal to a positive constant $|\mathbf{Q}|^{1/2} (k!)^{-1} (2\pi)^{-n/2}$ times the integral

$$(4) \qquad \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} \left(-\tfrac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} x_i x_j q_{ij} \right)^k \exp\left\{ -\tfrac{1}{2} \sum_{i=1}^{n} x_i^2 q_{ii} \right\} \mathrm{d}x_1 \ldots \mathrm{d}x_n\,.$$

The evaluation of this integral (4) is described on p. 175 in [1], and it turns out that it equals a sum of certain products of the type

$$(5) \qquad \varrho(\alpha_1, \ldots, \alpha_n) = C \prod_{i=1}^{n} \int_{-a_i}^{a_i} x_i^{\alpha_i} \exp\{-\tfrac{1}{2} x_i^2 q_{ii}\} \, \mathrm{d}x_i\,,$$

where C is a constant, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are even numbers. Therefore, obviously, all integrals in (5) are positive. As for the constant C, it is easily seen to be equal to the product of a certain polynomial coefficient and of some numbers $-\frac{1}{2} q_{ij}$ (with $i \neq j$); since $q_{ij} \leqq 0$ for $i \neq j$ by formula (3), we have $-\frac{1}{2} q_{ij} \geqq 0$, and the proof is finished.

It would be interesting to find a general class of covariance matrices $\mathbf{G}$ for which $q_{ij} \leqq 0$ $(i \neq j)$, giving thus $c_k \geqq 0$. In this connection it may be remarked that the matrix $\mathbf{Q}$ in our present case, being positive definite, belongs to the class $\mathbf{K}$ of matrices with non-positive off-diagonal elements and positive principal minors studied by Fiedler and Pták [5]. However, it seems that among many results in [5] only one has a consequence related to our present problem: namely, by Theorem (4,3) 11° in [5] it follows that $\mathbf{Q}$ positive definite, $q_{ij} \leqq 0$ for $i \neq j$, imply that $\mathbf{G}$ must have only non-negative elements.

Next, let us find the smallest characteristic root λ_1 of the matrix $\mathbf{Q}$, which is needed in Assertion 1. If either all g_i's, or all g_i's but one, are equal to 0, then clearly $\mathbf{G} = \mathbf{I}$ (the unit matrix), $\mathbf{Q} = \mathbf{I}$, $\lambda_1 = 1$. Therefore let us assume from now on that the number of positive g_i's is at least 2.

Assertion 6. *If $\mathbf{G}$ satisfies the above assumptions, $\lambda_1 = 1/(z_0 + 1)$ where z_0 is the unique positive solution of the equation*

$$\sum_{i=1}^{n} \frac{g_i^2}{z + g_i^2} = 1 \,. \tag{6}$$

Proof. First, the function $\psi(z) = \sum_{i=1}^{n} g_i^2(z + g_i^2)^{-1}$ is decreasing and continuous for $0 < z < \infty$; morever, $\lim_{z \to 0+} \psi(z)$ is equal to the number of positive g_i's, i.e. at least 2, and $\lim_{z \to \infty} \psi(z) = 0$, so that there exists a unique positive solution z_0 of $\psi(z) = 1$, i.e. of the equation (6).

Second, (6) for z_0 implies evidently

$$(z_0 + g_k^2) \sum_{i=1}^{n} \frac{g_i^2}{z_0 + g_i^2} - g_k^2 = z_0 \quad \text{for each} \quad k = 1, \ldots, n \,, \tag{7}$$

or equivalently,

$$(z_0 + g_k^2) \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{g_i^2}{z_0 + g_i^2} = z_0 \quad \text{for each} \quad k = 1, \ldots, n \,. \tag{8}$$

On the other hand, consider now the matrix $\mathbf{H} = \mathbf{G} - \mathbf{I}$ (i.e. the matrix with diagonal elements 0, off-diagonal elements $g_i g_j$), and let its roots be $\chi_1 \leqq \chi_2 \leqq \ldots \ldots \leqq \chi_n$. Since $\mathbf{H}$ is non-negative, by a well known result (see e.g. [6], formula (41) at the end of Chapter XIII, § 2) we have for its maximal root χ_n the inequalities

$$\min_{1 \leqq k \leqq n} \frac{(\mathbf{H}\mathbf{x})_k}{x_k} \leqq \chi_n \leqq \max_{1 \leqq k \leqq n} \frac{(\mathbf{H}\mathbf{x})_k}{x_k} \quad \text{for each} \quad \mathbf{x} \geqq 0 \,, \quad \mathbf{x} \neq 0 \,. \tag{9}$$

Insert now into (9) the column vector $\mathbf{x}$ with coordinates

$$x_1 = g_1(z_0 + g_1^2)^{-1}, \; x_2 = g_2(z_0 + g_2^2)^{-1}, \ldots, x_n = g_n(z_0 + g_n^2)^{-1} \,.$$

We get then

$$\frac{(\mathbf{H}\mathbf{x})_k}{x_k} = \frac{\sum_{i \neq k} g_i g_k \cdot g_i (z_0 + g_i^2)^{-1}}{g_k (z_0 + g_k^2)^{-1}} = \sum_{i \neq k} \frac{g_i^2 (z_0 + g_k^2)}{z_0 + g_i^2} = z_0 \quad \text{for each} \quad k = 1, \ldots, n \,,$$

where the last equality follows from (8). Hence (9) gives $z_0 \leqq \chi_n \leqq z_0$, i.e. $\chi_n = z_0$.

Further, $\boldsymbol{H}\boldsymbol{u} = \chi\boldsymbol{u}$ implies $\boldsymbol{G}\boldsymbol{u} = (\boldsymbol{H} + \boldsymbol{I})\,\boldsymbol{u} = \chi\boldsymbol{u} + \boldsymbol{u} = (\chi + 1)\,\boldsymbol{u}$ and conversely, so that $\gamma_n = \chi_n + 1 = z_0 + 1$. Finally, we have clearly $\lambda_1 = 1/\gamma_n = 1/(z_0 + 1)$.

Assertion 7. *We have the following inequalities:*

$$\sum_{i=1}^{n} g_i^2 - \max_{1 \leqq k \leqq n} g_k^2 \leqq z_0 \leqq \sum_{i=1}^{n} g_i^2 - \min_{1 \leqq k \leqq n} g_k^2 \,, \tag{10}$$

$$\min_{1 \leqq k \leqq n} g_k \sum_{i \neq k} g_i \leqq z_0 \leqq \max_{1 \leqq k \leqq n} g_k \sum_{i \neq k} g_i \,. \tag{11}$$

Proof. If we put $\psi_1(z) = \sum_{i=1}^{n} g_i^2(z + \max_{1 \leqq k \leqq n} g_k^2)^{-1}$, we have obviously $\psi_1(z) \leqq \psi(z)$. Hence the solution z_1 of $\psi_1(z) = 1$ must satisfy $z_1 \leqq z_0$, but z_1 is the left hand side of (10). The right hand side of (10) is obtained similarly on using min g_k^2 in place of max g_k^2 in the preceding argument.

The inequalities (11) are obtained immediately from (9) on inserting there the vector $\boldsymbol{x}$ with coordinates 1, 1, ..., 1. The proof is finished.

It may be observed that sometimes the inequalities (10), but sometimes the inequalities (11) give closer results.

The inequalities (10) and (11) may be used for two purposes: either, the bounds in them may serve as first approximations for z_0 in a numerical solution of (6); or, since $\gamma_n = z_0 + 1$, the upper bounds in (10) and (11) plus one may serve as γ in Assertion 2.

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Souhrn

POZNÁMKY K ANDĚLOVU ČLÁNKU "ON MULTIPLE NORMAL PROBABILITIES OF RECTANGLES"

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Předkládá se několik drobných poznámek k předcházejícímu článku J. Anděla [1]. V první části se uvádějí některá zjednodušení pro číslo m, které se vyskytuje v hranici Z_t pro zbytek řady navržené v [1]. Druhá část je věnována speciálnímu případu matic $\boldsymbol{G}$, pro něž $g_{11} = g_{22} = \ldots = g_{nn} = 1$; $g_{ij} = g_i g_j$ pro $i \neq j$; $i, j = 1, \ldots, n$, kde $0 \leqq g_i < 1$ pro $i = 1, \ldots, n$; zvláště je ukázáno, že řada navržená v [1] má pro případ takovýchto matic pouze nezáporné členy.

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