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WEIERSTRASS $\wp$-FUNCTION

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The solution of a motion of the rigid body with one fixed point leads to higher transcendental functions. We are publishing two papers on this problem by the late Prof. J. Chrapan.

According to the definition [1, p. 153, line 2]

$$\wp(u) \equiv \wp_1(u) = \frac{1}{12\omega^2}\frac{\vartheta_1'''}{\vartheta_1'} - \frac{d^2}{du^2}\ln \vartheta_1\left(\frac{u}{2\omega}\right) \tag{1}$$

and with respect to the relation

$$\frac{\vartheta_1'''}{\vartheta_1'} = \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_0''}{\vartheta_0} \tag{2}$$

[1, p. 142, line 4 from below] let us introduce the functions

$$\wp_0(u) = \frac{1}{4\omega^2}\frac{\vartheta_0''}{\vartheta_0} - \frac{d^2}{du^2}\ln \vartheta_0\left(\frac{u}{2\omega}\right); \tag{3}$$

$$\wp_2(u) = \frac{1}{4\omega^2}\frac{\vartheta_2''}{\vartheta_2} - \frac{d^2}{du^2}\ln \vartheta_2\left(\frac{u}{2\omega}\right);$$

$$\wp_3(u) = \frac{1}{4\omega^2}\frac{\vartheta_3''}{\vartheta_3} - \frac{d^2}{du^2}\ln \vartheta_3\left(\frac{u}{2\omega}\right).$$

Differentiating the relations [2, p. 243, (1) and (2)] we obtain

$$\frac{d}{dv} Z_\alpha(v; k) = Z_\alpha'(v; k) = \frac{d^2}{dv^2}\ln \vartheta_\alpha\left(\frac{v}{2K}; i\frac{K'}{K}\right) = \frac{\Theta_\alpha''(v; k)}{\Theta_\alpha(v; k)} - Z_\alpha^2(v; k), \tag{4}$$

where the index $\alpha = 0, 1, 2, 3$; K, K' are complete elliptic integrals of the first type

[1, p. 142, § 3] and k is the modulus of Jacobi elliptic functions. With regard to the relations

$$(5) \qquad \frac{v}{2K} = \frac{u}{2\omega}\,; \quad i\,\frac{K'}{K} = \frac{\omega'}{\omega}\,,$$

where ω, ω' are the half-periods of the Weierstrass $\wp$-function (1) [1, p. 151, § 5], formula (4) yields

$$(6) \qquad \frac{\mathrm{d}^2}{\mathrm{d}u^2} \ln \vartheta_\alpha\left(\frac{u}{2\omega}\right) = \frac{K^2}{\omega^2}\, Z'_\alpha(v; k)$$

and, for Θ-functions of zero arguments

(7)

$$\frac{\Theta''_0}{\Theta_0} = \frac{1}{4K^2}\frac{\vartheta''_0}{\vartheta_0} = 1 - \frac{E}{K}\,; \quad \frac{\Theta''_2}{\Theta_2} = \frac{1}{4K^2}\frac{\vartheta''_2}{\vartheta_2} = -\frac{E}{K}\,; \quad \frac{\Theta''_3}{\Theta_3} = \frac{1}{4K^2}\frac{\vartheta''_3}{\vartheta_3} = k'^2 - \frac{E}{K}$$

hence with regard to (2)

$$(8) \qquad \frac{\vartheta'''_1}{\vartheta'_1} = 12K^2\left[\frac{1}{3}(1 + k'^2) - \frac{E}{K}\right],$$

where k' is the complementary modulus od Jacobi elliptic functions and E is the complete elliptic integral of the second type.

Substituting expressions (6), (7) and (8) into (1) and (2) we obtain

$$(9) \qquad \wp_0(u) = \frac{K^2}{\omega^2}\left[1 - \frac{E}{K} - Z'_0(v; k)\right];$$

$$\wp_1(u) = \frac{K^2}{\omega^2}\left[\frac{1}{3}(1 + k'^2) - \frac{E}{K} - Z'_1(v; k)\right];$$

$$\wp_2(u) = \frac{K^2}{\omega^2}\left[-\frac{E}{K} - Z'_2(v; k)\right];$$

$$\wp_3(u) = \frac{K^2}{\omega^2}\left[k'^2 - \frac{E}{K} - Z'_3(v; k)\right],$$

or, with respect to [2, p. 245, (5)]

$$(10) \qquad \wp_0(u) = \frac{K^2}{\omega^2}\, k^2\, sn^2(v; k)\,;$$

$$\wp_1(u) = -\frac{K^2}{\omega^2}\left[\tfrac{1}{3}(1 + k^2) - ns^2(v; k)\right];$$

$$\wp_2(u) = \frac{K^2}{\omega^2} k'^2 \, sc^2(v; k) \, ;$$

$$\wp_3(u) = -\frac{K^2}{\omega^2} k^2 k'^2 \, sd^2(v; k) \, .$$

Formulae (10) yield the following values:

u	$\wp_0(u)$	$\wp_1(u)$	$\wp_2(u)$	$\wp_3(u)$
0	0	∞	0	0
$\frac{1}{2}\omega$	$\frac{K^2}{\omega^2}(1 - k')$	$\frac{K^2}{\omega^2}[k' + \frac{1}{3}(1 + k'^2)]$	$\frac{K^2}{\omega^2} k'$	$-\frac{K^2}{\omega^2} k'(1 - k')$
ω	$\frac{K^2}{\omega^2} k^2$	$\frac{1}{3}\frac{K^2}{\omega^2}(1 + k'^2)$	∞	$-\frac{K^2}{\omega^2} k^2$
$\frac{3}{2}\omega$	$\frac{K^2}{\omega^2}(1 - k')$	$\frac{K^2}{\omega^2}[k' + \frac{1}{3}(1 + k'^2)]$	$\frac{K^2}{\omega^2} k'$	$-\frac{K^2}{\omega^2} k'(1 - k')$
2ω	0	∞	0	0
$\frac{1}{2}\omega'$	$\frac{K^2}{\omega^2} k$	$-\frac{K^2}{\omega^2}[k + \frac{1}{3}(1 + k^2)]$	$-\frac{K^2}{\omega^2}(1 - k)$	$\frac{K^2}{\omega^2} k(1 - k)$
ω'	∞	$-\frac{1}{3}\frac{K^2}{\omega^2}(1 + k^2)$	$-\frac{K^2}{\omega^2} k'^2$	$\frac{K^2}{\omega^2} k'^2$
$\frac{3}{2}\omega'$	$-\frac{K^2}{\omega^2} k$	$-\frac{K^2}{\omega^2}[k + \frac{1}{3}(1 + k^2)]$	$-\frac{K^2}{\omega^2}(1 - k)$	$\frac{K^2}{\omega^2} k(1 - k)$
$2\omega'$	0	∞	0	0
$\omega + \omega'$	$\frac{K^2}{\omega^2}$	$-\frac{1}{3}\frac{K^2}{\omega^2}(1 - 2k^2)$	$-\frac{K^2}{\omega^2}$	∞

$$\wp_1(\omega) = e_1 \, ; \quad \wp_1(\omega + \omega') = e_2 \, ; \quad \gamma_1(\omega') = e_3$$

are zero points of the cubic polynomial of the function (1) [1, p. 152, line 12].

The results given in this table enable us to establish, with regard to (10), the following relations:

u	$\wp_0(u)$	$\wp_1(u)$	$\wp_2(u)$	$\wp_3(u)$
$w+\omega$	$\wp_3(w)+\wp_0(\omega)=$ $=\wp_3(w)-\wp_3(\omega)$	$\wp_2(w)+\wp_1(\omega)$	$\wp_1(w)-\wp_1(\omega)$	$\wp_0(w)+\wp_3(\omega)=$ $=\wp_0(w)-\wp_0(\omega)$
$w+2\omega$	$\wp_0(w)$	$\wp_1(w)$	$\wp_2(w)$	$\wp_3(w)$
$w+\omega'$	$\wp_1(w)-\wp_1(\omega')$	$\wp_0(w)+\wp_1(\omega')$	$\wp_3(w)+\wp_2(\omega')=$ $=\wp_3(w)-\wp_3(\omega')$	$\wp_2(w)+\wp_3(\omega')=$ $=\wp_2(w)-\wp_2(\omega')$
$w+2\omega'$	$\wp_0(w)$	$\wp_1(w)$	$\wp_2(w)$	$\wp_3(w)$
$w+\omega+$ $+\omega'$	$\wp_2(w)+$ $+\wp_0(\omega+\omega')=$ $=\wp_2(w)-$ $-\wp_2(\omega+\omega')$	$\wp_3(w)+$ $+\wp_1(\omega+\omega')$	$\wp_0(w)+$ $+\wp_2(\omega+\omega')=$ $=\wp_0(w)-$ $-\wp_0(\omega+\omega')$	$\wp_1(w)-$ $-\wp_1(\omega+\omega.)$

According to the identity $\wp_\alpha(w+2\omega)=\wp_\alpha(w+2\omega')=\wp_\alpha(w)$ expressions (1) and (3) are double periodic functions (of the first type) and with respect to (1) they define the Weierstrass transcendental functions ($\wp$-functions). It follows from (10) that these functions are even: $\wp_\alpha(-u)=\wp_\alpha(u)$.

References

[1] *W. Magnus, F. Oberhettinger:* Formeln und Sätze für die speziellen Funktionen der mathematischen Physik. Berlin—Göttingen—Heidelberg: Springer-Verlag 1948.

[2] *J. Chrapan:* Matematicko-fyzikálny časopis SAV 4 (1961), 244.

Súhrn

WEIERSTRASSOVE $\wp$-FUNKCIE

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Uvádzajú se Weierstrassove péfunkcie a v dvoch tabuľkách sa formulujú ich význačné hodnoty a vzťahy medzi nimi.

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