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EVALUATION OF THE HALF-PERIODS OF THE WEIERSTRASS  
 $\wp$ -FUNCTION FOR THE ABSOLUTE INVARIANT GREATER THAN ONE

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*The paper is a continuation of that on Weierstrass  $\wp$ -functions by the same author [3]. The solution of this mathematical problem is connected with a motion of a rigid body with one fixed point.*

The half-periods of the Weierstrass  $\wp$ -function [1, p. 328; 13.12] may be determined by means of relations [1, p. 341; (9)]:

$$(1) \quad \omega = \frac{K}{\sqrt{(e_1 - e_3)}};$$

$$(2) \quad \omega' = \frac{iK'}{\sqrt{(e_1 - e_3)}},$$

where  $K, K'$  are the constants of the periods of Jacobi elliptic functions (complete elliptic integrals of the first type),  $e_1, e_3$  are zero points of the Weierstrass cubic polynomial [1, p. 338; (10)] which have different real values provided that the absolute invariant [1, p. 375; (4), (5)] is greater than one:

$$(3) \quad J = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{4(1 - k^2 + k^4)^3}{27k^4k'^4} = \frac{4(1 - k^2k'^2)^3}{27k^4k'^4} > 1.$$

Considering (3) we obtain

$$\frac{g_3^2}{g_2^3} = \frac{1}{108} \frac{(1 + k^2)^2 (1 - 2k^2)^2 (2 - k^2)^2}{(1 - k^2k'^2)^3},$$

which yields a reciprocal equation of the sixth degree for the unknown  $k^2$ . The solutions of this equation fulfil the relation [1, p. 340; 13.16, (3)] for the six permutations of the zero points  $e_1, e_2, e_3$  of the Weierstrass cubic polynomial [1, p. 338; (10)].

After a modification and reduction we obtain the cubic equation

$$(4) \quad (a - 1)^3 - \frac{27}{4} J(a - 1) + \frac{27}{4} J = 0.$$

Let the zero points of the Weierstrass cubic polynomial [1, p. 338; (10)] fulfil the inequality

$$e_1 > e_2 > e_3,$$

so that according to [1, p. 342; (11)] the moduli of Jacobi elliptic functions [1, p. 340; 13.16] fulfil

$$0 < k^2 < 1; \quad 0 < k'^2 < 1.$$

According to [1, p. 340; 13.16, (3)]

$$e_1 - e_3 = 3\rho > 0$$

holds and with respect to [1, p. 332; (5)] the invariant

$$\begin{aligned} g_3 &= 4e_1e_2e_3 = 4\rho^3(2 - k^2)(1 - 2k^2)(1 + k^2) = \\ &= 4\rho^3(1 + k'^2)(2k'^2 - 1)(2 - k'^2), \end{aligned}$$

so that

$$\text{sign } g_3 = \text{sign}(1 - 2k^2) = \text{sign}(2k'^2 - 1).$$

Consequently, if  $g_3 > 0$ , then the inequalities

$$(5) \quad 0 < k^2 < \frac{1}{2}; \quad \frac{1}{2} < k'^2 < 1;$$

hold; if  $g_3 < 0$ , then

$$\frac{1}{2} < k^2 < 1; \quad 0 < k'^2 < \frac{1}{2}.$$

Hence the evaluation of the most suitable value of the modulus  $k$  under the condition (3) is given by the relation

$$(6) \quad [k^2]_{g_3 > 0} = \frac{a}{2} - \sqrt{\left[\left(\frac{a}{2}\right)^2 - 1\right]};$$

$$(7) \quad [k^2]_{g_3 < 0} = 1 - \frac{a}{2} + \sqrt{\left[\left(\frac{a}{2}\right)^2 - 1\right]},$$

where, using the goniometric solution of (4), we have

$$(8) \quad a = 1 + 3 \frac{\cos \frac{1}{3}(\pi - \varphi)}{\cos \varphi} > \frac{5}{2};$$

$$(9) \quad \cos \varphi = \frac{1}{\sqrt{J}};$$

$$(10) \quad 0 < \varphi < \frac{\pi}{2}.$$

With respect to [1, p. 340; 13.16, (3) and p. 332; (5), (6) respectively], the half-periods (1) and (2) if the Weierstrass  $\wp$ -functions are given by

$$(11) \quad \begin{aligned} \omega &= K \sqrt{2} \sqrt[4]{\left(\frac{1 - k^2 k'^2}{3g_2}\right)} = \\ &= K \frac{\sqrt[3]{2}}{\sqrt{3}} \sqrt[6]{\left(\frac{(1 + k^2)(1 - 2k^2)(2 - k^2)}{g_3}\right)}; \end{aligned}$$

$$(12) \quad \begin{aligned} \omega' &= iK' \sqrt{2} \sqrt[4]{\left(\frac{1 - k^2 k'^2}{3g_2}\right)} = \\ &= iK' \frac{\sqrt[3]{2}}{\sqrt{3}} \sqrt[6]{\left(\frac{(1 + k^2)(1 - 2k^2)(2 - k^2)}{g_3}\right)} \end{aligned}$$

provided condition (3) is fulfilled.

Examples: 1. Consider the differential equation

$$\left(\frac{1}{2} \frac{dy}{dx}\right)^2 = y^3 - 24y - 16.$$

After a modification we obtain the equation

$$(13) \quad \left(\frac{dy}{dx}\right)^2 = 4y^3 - 96y - 64,$$

which is satisfied by the Weierstrass function

$$(14) \quad y = \wp(x + c),$$

$c$  being a constant of integration [1, p. 332; (4)]. Let us find the half-periods (1), (2) of the Weierstrass function (14).

According to (13)  $g_2 = 96$ ;  $g_3 = 64$ , hence the absolute invariant

$$J = 1,142857 \dots > 1.$$

Substituting this value into (9) we obtain with respect to (10)

$$\varphi = 0,361367123 \dots,$$

which implies with regard to (8)

$$a = 2,92570 \dots$$

and, according to (6) and (5)

$$k^2 = 0,39517 \dots,$$

Thus after the evaluation of complete integrals [2, p. 105–108]

$$K = 1,7741_6; \quad K' = 1,9547_9,$$

and with respect to (11) and (12) we obtain the half-periods (1) and (2) respectively:

$$\omega = 0,5688 \dots \quad \omega' = 0,6267 \dots i.$$

If it were  $g_3 = -64$  in the equation (13), then according to (7) we should have  $k^2 = 0,60482 \dots$ . Hence according to (11) and (12) the half-periods (1) and (2) would be  $\omega = 0,6267 \dots$  and  $\omega' = 0,5688 \dots$  respectively.

2. If we have the differential equation

$$3 \left( \frac{3}{2} \frac{dy}{dx} \right)^2 = 27y^3 - 117y - 92,$$

we modify it to

$$(15) \quad \left( \frac{dy}{dx} \right)^2 = 4y^3 - \frac{52}{3}y - \frac{368}{27},$$

so that the invariants

$$g_2 = \frac{52}{3}; \quad g_3 = \frac{368}{27},$$

hence the absolute invariant

$$J = 27,12345 \dots > 1.$$

After the substitution of this value into (9) we obtain with regard to (10)

$$\varphi = 1,377585830 \dots,$$

which implies according to (8)

$$a = 14$$

and according to (6) and (5) it is

$$k^2 = 0,071796 \dots$$

Thus after the evaluation of complete elliptic integrals [2, p. 105–108]

$$K = 1,6001_8 ; K' = 2,7350_4 ,$$

and with respect to (11) and (12) we have the half-periods (1) and (2) respectively:

$$\omega = 0,8283 \dots \quad \omega' = 1,4157 \dots i .$$

If it were  $g_3 = -368/27$ , then according to (7) we should have  $k^2 = 0,92820 \dots$ . Hence according to (11) and (12) the half-periods (1) and (2) would be  $\omega = 1,4157 \dots$  and  $\omega' = 0,8283 \dots i$ .

#### References

- [1] *H. Bateman, A. Erdélyi: Higher Transcendental Functions. Volume II. New York—Toronto—London: McGraw-Hill Book Company, Inc. 1953.*
- [2] *K. Uhde: Spezielle Funktionen der mathematischen Physik, Tafeln II. Mannheim: Bibliographisches Institut 1964, 105–108.*
- [3] *J. Chrapan: Weierstrass  $\wp$ -function. Aplikace matematiky 4, 16 (1971).*

#### Súhrn

### VYČÍSLLENIE POLPERIÓD WEIERSTRASSOVEJ $\wp$ -FUNKCIE PRI ABSOLÚTNOM INVARIANTE VÄČŠOM AKO ČÍSLO 1

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V práci sú odvodené výrazy vhodné pre vyčíslenie polperiód Weierstrassovej  $\wp$ -funkcie pri absolútnom invariante väčšom ako číslo 1 a výpočet je ilustrovaný na dvoch numerických príkladoch.

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