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SOME PROPERTIES OF LINEAR HOMOGENEOUS TRANSFORMATION
OF INDEPENDENT VARIABLE IN ORDINARY
DIFFERENTIAL LINEAR EQUATIONS

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INTRODUCTION

Let \( \mathbb{C} \) denote the set of all complex numbers.

Let a self-adjoint equation of the 2nd order in the complex domain

\[
[\Theta(t) y'(t)]' - B(t) y(t) = 0
\]

be given.

Put \( z(t) = y(kt) \), where \( k \in \mathbb{C} \). Then \( z(t) \) satisfies the self-adjoint equation

\[
[\Theta(kt) z'(t)]' - k^2 B(kt) z(t) = 0 .
\]

Let \( \mathcal{L} y(t) \) be an arbitrary linear homogeneous differential operator of the 2nd order with the property (for some \( m \in \mathbb{C} \))

\[
\mathcal{L} y(kt) = k^m \mathcal{L} y(kt) .
\]

Let \( f(t) \neq 0 \) be an arbitrary factor. Put

\[
\mathcal{L} y(t) = f(t) \mathcal{L} y(t) .
\]

Then it holds

\[
\mathcal{L} y(kt) = \frac{f(t)}{f(kt)} k^m \mathcal{L} y(kt) .
\]

Hence it follows that the operator \( \mathcal{L} y(t) \) fulfils for some \( r \in \mathbb{C} \)

\[
\mathcal{L} y(kt) = k^r \mathcal{L} y(kt) .
\]
iff (= if and only if) the factor $f(t)$ is a homogeneous function of degree $m - r$, i.e.

\begin{equation}
(0.7) \quad f(kt) = k^{m-r} f(t).
\end{equation}

holds.

Let the self-adjoint operator of the 2nd order $Ly(t) = [\Theta(t) y'(t)]' - B(t) y(t)$ fulfill (0.3). Let $A(t)$ be a homogeneous function of degree $r$, i.e. $A(kt) = k^r A(t)$. If $y(t)$ is a solution of the equation

\begin{equation}
(0.8) \quad Ly(t) = -A(t) y(t),
\end{equation}

then $z(t) = y(kt)$ is a solution of the equation

\begin{equation}
(0.9) \quad [\Theta(t) z'(t)]' + [\lambda A(t) - B(t)] z(t) = 0,
\end{equation}

where $\lambda = k^{m+r}$.

1. Let us have a real self-adjoint equation in a suitable interval (which will be determined later)

\begin{equation}
(1.1) \quad [\Theta(t) y'(t)]' - B(t) y(t) = -A(t) y(t)
\end{equation}

where for suitable real numbers $m, r$ it is

(i) $\Theta(kt) = k^{2-m} \Theta(t)$, $\Theta \in C^1$, $\Theta(t) \neq 0$,

(ii) $B(kt) = k^{-m} B(t)$ or $B = 0$, $B \in C^0$,

(iii) $A(kt) = k^r A(t)$, $A(t) > 0$, $A \in C^0$.

For an arbitrary fixed non-trivial solution $y$ of the equation (1.1) let us put $z(t) = y(kt)$ where $k$ is a real parameter. Then for $\lambda = k^{m+r}$ the function $z(t)$ satisfies the differential equation

\begin{equation}
(1.2) \quad [\Theta(t) z'(t)]' + [\lambda A(t) - B(t)] z(t) = 0.
\end{equation}

If $m + r \neq 0$, $k > 0$, then $\lambda > 0$ and by the relation $\lambda = k^{m+r}$ a one-to-one correspondence is given between the values $k, \lambda$. Then $\lambda \uparrow \infty$ iff either $m + r > 0$ and $k \uparrow \infty$, or $m + r < 0$ and $k \downarrow 0$. (The symbols $\uparrow, \downarrow$ include the monotony of the convergence).

Suppose the equation (1.1), in case of $m + r > 0$, to be oscillatory (with infinitely many roots) in an annular neighborhood $O^*_\infty$ of the point $\infty$, while in case of $m + r < 0$ it is supposed to be oscillatory (with infinitely many roots) in an annular neighborhood $O^-_0$, of the point 0 from the right side.

Let $k_1, k_2, k_3, \ldots$ range monotonically over all positive roots of the solution $y$ of the equation (1.1) in such a way that the corresponding values $\lambda_1, \lambda_2, \lambda_3, \ldots$ are increasing to $\infty$. Put $z_i(t) = y(k_it), i = 1, 2, 3, \ldots$
I. The case $m + r > 0$. Let the equation (1.1) be defined in some interval $]a, \infty[$ where $0 \leq a < k_1$ so that $k_1$ is still an inner point of the domain of the equation (1.1). Then for $k \geq k_1$, $z(t) = y(kt)$ is defined in the interval $]b, 1[$ where $(0 \leq b = (1/k_1) a(<1)$, since for $k \geq k_1$ the transformation $t \rightarrow kt$ maps the interval $]b, 1[$ onto the interval $]kb, k[ \subseteq ]a, \infty[$.

II. The case $m + r < 0$. Let the equation (1.1) be defined in some interval $]0, a[$ where $0 < k < a \leq \infty$ so that $k_1$ is still an inner point of the domain of the equation (1.1). Put $(\infty ^\circ) b = (1/k) a(>1)$. For $0 < k \leq k_1$, the transformation $t \rightarrow kt$ maps the interval $[1, b[$ onto the interval $[k, kb[ \subseteq ]0, a[$ and thus the function $z(t) = y(kt)$ is defined in the interval $[1, b[$.

In the case I the function $z_i(t)$, $i = 1, 2, 3, \ldots$ satisfies the differential equation (1.2) with the parameter $\lambda = \lambda_i$ in the interval $]b, 1[$ and thus for $\beta \in ]b, 1[$ it holds

$$\lim_{\beta \to b^-} \Omega(\beta) \left[ z_i'(\beta) z_j(\beta) - z_i(\beta) z_j'(\beta) \right] = 0 .$$

Hence it follows that the sequence of functions $z_1, z_2, z_3, \ldots$ is orthogonal with the weight $A(t)$ iff for any $i \neq j$

$$\lim_{\beta \to b^-} \Omega(\beta) \left[ z_i'(\beta) z_j(\beta) - z_i(\beta) z_j'(\beta) \right] = 0 .$$

In the case II the function $z_i(t)$, $i = 1, 2, 3, \ldots$ satisfies the differential equation (1.2) with the parameter $\lambda = \lambda_i$ in the interval $[1, b[$, and thus for $\beta \in [1, b[$ it holds

$$\lim_{\beta \to b^-} \Omega(\beta) \left[ z_i'(\beta) z_j(\beta) - z_i(\beta) z_j'(\beta) \right] = 0 .$$

Hence it follows that the sequence of functions $z_1, z_2, z_3, \ldots$ is orthogonal with the weight $A(t)$ iff for any $i \neq j$

$$\lim_{\beta \to b^-} \Omega(\beta) \left[ z_i'(\beta) z_j(\beta) - z_i(\beta) z_j'(\beta) \right] = 0 .$$

Example. For the Bessel equation

$$(ty')' - \frac{n^2}{t} y = -ty , \quad n \geq 0 \quad \text{fixed}$$

in the interval $]0, \infty[$ we have $\Omega(t) = t$, $B(t) = n^2/t$, $A(t) = t$, $m = 1$, $r = 1$. In the interval $]0, \infty[$ the solution $J_n(t)$ has infinitely many roots $k_i$, $i = 1, 2, 3, \ldots$ increasing to $\infty$ so that the case I occurs. The functions $J_n(k_i t)$, $i = 1, 2, 3, \ldots$ form in the interval $]0, 1[$ an orthogonal sequence with the weight $t$ iff for $i \neq j$

$$\lim_{t \to 0^+} \left\{ t \left[ k_i J_n'(k_i t) J_n(k_i t) - k_j J_n'(k_i t) J_n(k_i t) \right] \right\} = 0 .$$
According to the formula \( J'_n(x) = -J_{n+1}(x) + (n/x) J_n(x) \), the expression \( p(t) \) following the limit symbol in (1.8) is reduced to \( p(t) = p_1(t) - p_2(t) \) where \( p_1(t) = t k_j J_{n+1}(k_j t) J_n(k_j t) \).

Consider that the following two rules hold for the asymptotic equality \( \sim \):

1° \( a_1 \sim b_1, \ a_2 \sim b_2 \), \( i = 1, 2 \Rightarrow a_1 a_2 \sim b_1 b_2 \),

2° \( a \sim b, \ b \sim 0 \Rightarrow a \sim 0 \).

From the formula \( J_n(x) \sim x^n/(2^n \Gamma(1 + n)) \) for \( x \to 0 \) we have then

\[
p_1(t) \sim \frac{k_j}{2^{2n+1} \Gamma(2 + n) \Gamma(1 + n)} t^{2(n+1)},
\]

\[
p_2(t) \sim \frac{k_i}{2^{2n+1} \Gamma(2 + n) \Gamma(1 + n)} t^{2(n+1)}
\]

so that \( p_i(t) \to 0, \ i = 1, 2 \) holds and thus \( p(t) \to 0 \) for \( t \to 0 \) iff \( n + 1 > 0 \).

2. Let the linear differential operator of the \( n \)-th order in the complex domain

\[
(L y(t) = \sum_{i=1}^{n} a_i(t) y^{(i)}(t)
\]

have the following property: After the linear substitution \( t \to k t, \ k \in \mathbb{C} \), it fulfils for a suitable \( m \in \mathbb{C} \) the relation

\[
(L y(k t) = k^m L y(k t).
\]

Let the differential equation

\[
(L y(t) = \lambda y(t)
\]

have a solution \( y(t) \) for a constant \( \lambda \in \mathbb{C} \). Then the function \( y(k t) \) satisfies the equation

\[
(L y(k t) = \lambda k^m y(k t).
\]

Form an equation of the \( 2n \)-th order with the operator \( L^2 = L^2 L \) and constants \( p, q \in \mathbb{C} \)

\[
(L^2 y(t) + 2 p L y(t) + q y(t) = 0.
\]

Look for its solution in the form \( y(k t) \) where \( y(t) \) is a solution of (2.3) and \( k \) is a suitable constant. We get „a characteristic” equation for the unknown \( k \)

\[
(\lambda k^m)^2 + 2p(\lambda k^m) + q = 0.
\]

For any \( k \) fulfilling (2.6) and for any \( y(t) \) fulfilling (2.3) the function \( y(k t) \) then fulfils (2.5).
In case of \( p^2 = q \) the equation (2.6) has a double root \( k^m = -p/\lambda \). The corresponding differential equation

\[
(\lambda + p)^2 y(t) = 0
\]

has a solution \( y(t) \) iff \( y(t) \) satisfies the equation

\[
(\lambda + p) y(t) = z(t)
\]

where \( z(t) \) is a suitable solution of the equation

\[
(\lambda + p) z(t) = 0.
\]

The last mentioned assertions hold generally for any operator \( A : M \to M \) on any set \( M \): for \( b \in M \) the equation \( A^2 y = b \) is equivalent to the equations \( Ay = z \), \( Az = b \).

3. For arbitrary \( n \in \mathbb{C} \) put, in the complex domain,

\[
\hat{n}E y(t) = y''(t) + \frac{1}{t} y'(t) - \frac{n^2}{t^2} y(t).
\]

Then the Bessel equation of the index \( n \) may be written in the form

\[
\hat{n}E y(t) = -y(t)
\]

or

\[
(\hat{n}E + 1) y(t) = 0.
\]

For an arbitrary \( k \in \mathbb{C} \) and for an arbitrary solution \( y(t) \) of the equation (3.3) the function \( z(t) = y(kt) \) is a solution of the equation

\[
(\hat{n}E + k^2) z(t) = 0.
\]

as the operator \( \hat{n}E y(t) \) has the property (2.2) for \( m = 2 \). From this property it also follows that, if \( y(t) \) is a solution of the equation

\[
(\hat{n}E + 1) y(t) = f(t)
\]

where \( f \) is an arbitrary continuous function, then for arbitrary \( k \in \mathbb{C} \) the function \( z(t) = y(kt) \) is a solution of the equation

\[
(\hat{n}E + k^2) z(t) = k^2 f(kt).
\]

Consider the iterated equation \((p, q \in \mathbb{C})\)

\[
(\hat{n}E^2 + 2p \hat{n}E + q) y(t) = 0.
\]
In case of \( y(t) \) being a solution of the Bessel equation (3.3), \( y(kt) \) is a solution of the equation (3.7) iff

\[ k_{1,2}^2 = p \pm \sqrt{(p^2 - q)}. \]

Combinations of the four values \( \pm k_1, \pm k_2 \in \mathbb{C} \) and of the two linearly independent solutions \( J_n(t), Y_n(t) \) of the equation (3.3) yield eight solutions of the equation (3.7). Since it holds for \( m \in \mathbb{Z}, n \in \mathbb{C} \) (\( \mathbb{Z} \) is the set of all integers)

\[ J_n(te^{im\pi}) = e^{im\pi} J_n(t), \]
\[ Y_n(te^{im\pi}) = e^{-im\pi} Y_n(t) + 2i \frac{m\pi n}{\sin n\pi} \cos n\pi J_n(t), \]

we can cancel the four solutions containing the arguments \(-k_1t, -k_2t, k_1t, k_2t\), because they are linear combinations of the others. The remaining solutions \( J_n(k_1t), J_n(k_2t), Y_n(k_1t), Y_n(k_2t) \) are linearly independent iff \( k_1 \neq k_2 \).

**Proof.** Take \( aJ_n(k_1t) + bJ_n(k_2t) + cY_n(k_1t) + dY_n(k_2t) = 0 \). Put \( y(t) = aJ_n(k_1t) + cY_n(k_1t) = -bJ_n(k_2t) - dY_n(k_2t) \). Then \( y(t) \) is a solution of the equation (3.4) for \( k = k_1 \) and \( k = k_2 \) so that \( k_1^2 y(t) = k_2^2 y(t) \). Hence in case of \( k_1 \neq k_2 \) we have \( y(t) = 0 \) and then \( a = c = 0, b = d = 0 \), Q.E.D.

In case of \( k_1 = k_2 = k \), i.e. by \( p^2 = q \), we get only two linearly independent solutions \( J_n(kt), Y_n(kt) \) of the equation (3.7), which is now of the form

\[ (\xi E + p)^2 y(t) = 0. \]

Since \( k^2 = p \) it is

\[ (\xi E + k^2)^2 y(t) = 0. \]

Let \( a, b \in \mathbb{C} \) be arbitrary fixed constants. Then the function \( Z_n(t) = aJ_n(t) + bY_n(t) \) is called a „general“ cylindrical function of the index \( n \). Since it is a fixed linear combination of the functions \( J_n(t), Y_n(t) \) with coefficients independent of the index \( n \), the same recurrent relations hold for \( Z_n(t) \) as for \( J_n(t) \) and \( Y_n(t) \), e.g.

\[ tZ_n'(t) - nZ_n(t) = -tZ_{n+1}(t). \]

The Bessel equation of the index \( n \) in the self-adjoint form is

\[ \left( t y' \right)' + \left( t - \frac{n^2}{t} \right) y = 0. \]

Put \( y(t) = tZ_{n+1}(t), [2] \). From the relation (3.13) we get

\[ y(t) = nZ_n(t) - tZ_n'(t). \]
Differentiating, multiplying by \( t \) and differentiating once more we get

\[(3.16) \quad (ty')' = n(tZ_n')' + (t^2 - n^2)Z_n' + 2tZ_n,\]

and once more by (3.13) we find

\[(3.17) \quad (ty')' + \left( t - \frac{n^2}{t} \right) y = 2tZ_n\]
or

\[(3.18) \quad (\mathcal{E} + 1) y(t) = 2Z_n(t).\]

From the considerations concerning (2.8) it appears that \( y(t) = tZ_{n+1}(t) \) is a solution of the equation

\[(3.19) \quad (\mathcal{E} + 1)^2 y(t) = 0.\]

According to (3.5), (3.6) it follows from (3.18) that the function \( z(t) = y(kt) \) is a solution of the equation

\[(3.20) \quad (\mathcal{E} + k^2) z(t) = 2k^2 Z_n(kt)\]

so that \( z(t) = y(kt) = ktZ_{n+1}(kt) \) satisfies the equation (3.12). So we find that the functions \( tJ_{n+1}(kt), tY_{n+1}(kt) \) are again solutions of the equation (3.12). At the same time the solutions \( J_n(kt), Y_n(kt), tJ_{n+1}(kt), tY_{n+1}(kt) \) of the equation (3.12) are linearly independent.

**Proof.** Consider a linear relation

\[a J_n(kt) + b Y_n(kt) + c J_{n+1}(kt) + d Y_{n+1}(kt) = 0.\]

Put \( Z_n(t) = -(c/k) J_n(t) - (d/k) Y_n(t) \). Then the function \( ktZ_{n+1}(kt) = -ct J_{n+1}(kt) - dt Y_{n+1}(kt) = a J_n(t) + b Y_n(t) \) is a solution of both equations (3.20) and (3.12). Hence it follows that \( Z_n = 0 \) so that \( c = d = 0 \) as well as \( a = b = 0 \); Q.E.D.

**References**


Souhrn

NĚKTERÉ VLASTNOSTI LINEÁRNÍ HOMOGENNÍ TRANSFORMACE NEZÁVISLE PROMĚNNÉ V OBYČEJNÝCH DIFERENCIÁLNÍCH LINEÁRNÍCH ROVNICÍCH

Erich Barvínek

Odst. 1 obsahuje obecnou lineární diferenciální rovnici 2. řádu (1.1), jejíž řešení $y(t)$ vytváří orthogonální posloupnost $y(k,t)$, kde $k_i$ je vhodně uspořádaná posloupnost kladných kořenů řešení $y(t)$. Jde v podstatě o „Eulerovské“ rovnice.


$$\left(\frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt}\right)w - 2b_0 \left(\frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt}\right)w + w = 0$$

v tom smyslu, že je nalezeno obecné řešení rovnice (3.7).

Pozoruhodná věta: je-li $Z_n(t) = a J_n(t) + b Y_n(t)$ libovolné řešení (3.3), $k \in \mathbb{C}$, libovolné, pak

1° $Z_n(kt)$ je řešení (3.4),

2° $kt Z_{n+1}(kt)$ je řešení (3.20) a tudíž (3.12),

je rozšířením úvah [2] o nalezení řešení $t Z_{n+1}(t)$ rovnice (3.12) a skýtá důkaz lineární nezávislosti jejich řešení $J_n(kt), Y_n(kt), t J_{n+1}(kt), t Y_{n+1}(kt)$.

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