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_Aplikace matematiky_, Vol. 16 (1971), No. 6, 431--438

Persistent URL: http://dml.cz/dmlcz/103378

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SOME PROPERTIES OF LINEAR HOMOGENEOUS TRANSFORMATION OF INDEPENDENT VARIABLE IN ORDINARY DIFFERENTIAL LINEAR EQUATIONS

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(Received October 26, 1970)

INTRODUCTION

Let \( C \) denote the set of all complex numbers.

Let a self-adjoint equation of the 2nd order in the complex domain

\[ [\Theta(t) y'(t)]' - B(t) y(t) = 0 \]

be given.

Put \( z(t) = y(kt) \), where \( k \in C \). Then \( z(t) \) satisfies the self-adjoint equation

\[ [\Theta(kt) z'(i)]' - k^2 B(kt) z(i) = 0 . \]

Let \( Ly(t) \) be an arbitrary linear homogeneous differential operator of the 2nd order with the property (for some \( m \in C \))

\[ Ly(kt) = k^m Ly(kt) . \]

Let \( f(t) \neq 0 \) be an arbitrary factor. Put

\[ Ly(t) = f(t) Ly(t) . \]

Then it holds

\[ Ly(kt) = \frac{f(t)}{f(kt)} k^m Ly(kt) . \]

Hence it follows that the operator \( Ly(t) \) fulfils for some \( r \in C \)

\[ Ly(kt) = k^r Ly(kt) . \]
iff (= if and only if) the factor $f(t)$ is a homogeneous function of degree $m - r$, i.e.

$$f(kt) = k^{m-r} f(t).$$

holds.

Let the self-adjoint operator of the 2nd order $Ly(t) = [\Theta(t) y'(t)]' - B(t) y(t)$ fulfill (0.3). Let $A(t)$ be a homogeneous function of degree $r$, i.e. $A(kt) = k^r A(t)$. If $y(t)$ is a solution of the equation

$$Ly(t) = -A(t) y(t),$$

then $z(t) = y(kt)$ is a solution of the equation

$$\left[\Theta(t) z'(t)\right]' + \left[\lambda A(t) - B(t)\right] z(t) = 0,$$

where $\lambda = k^{m+r}$.

1. Let us have a real self-adjoint equation in a suitable interval (which will be determined later)

$$\left[\Theta(t) y'(t)\right]' - B(t) y(t) = -A(t) y(t)$$

where for suitable real numbers $m, r$ it is

(i) $\Theta(kt) = k^{2-m} \Theta(t), \quad \Theta \in C^1, \quad \Theta(t) \neq 0,$

(ii) $B(kt) = k^{-m} B(t)$ or $B = 0, \quad B \in C^0,$

(iii) $A(kt) = k^r A(t), \quad A(t) > 0, \quad A \in C^0$.

For an arbitrary fixed non-trivial solution $y$ of the equation (1.1) let us put $z(t) = y(kt)$ where $k$ is a real parameter. Then for $\lambda = k^{m+r}$ the function $z(t)$ satisfies the differential equation

$$\left[\Theta(t) z'(t)\right]' + \left[\lambda A(t) - B(t)\right] z(t) = 0.$$

If $m + r \neq 0, k > 0$, then $\lambda > 0$ and by the relation $\lambda = k^{m+r}$ a one-to-one correspondence is given between the values $k, \lambda$. Then $\lambda \uparrow \infty$ iff either $m + r > 0$ and $k \uparrow \infty$, or $m + r < 0$ and $k \downarrow 0$. (The symbols $\uparrow, \downarrow$ include the monotony of the convergence).

Suppose the equation (1.1), in case of $m + r > 0$, to be oscillatory (with infinitely many roots) in an annular neighborhood $O^*_{\infty}$ of the point $\infty$, while in case of $m + r < 0$ it is supposed to be oscillatory (with infinitely many roots) in an annular neighborhood $O^*_0$, of the point 0 from the right side.

Let $k_1, k_2, k_3, \ldots$ range monotonically over all positive roots of the solution $y$ of the equation (1.1) in such a way that the corresponding values $\lambda_1, \lambda_2, \lambda_3, \ldots$ are increasing to $\infty$. Put $z_i(t) = y(k_i t), i = 1, 2, 3, \ldots$. 

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I. The case $m + r > 0$. Let the equation (1.1) be defined in some interval $]a, \infty[ \,$ where $0 \leq a < k_1$ so that $k_1$ is still an inner point of the domain of the equation (1.1). Then for $k \geq k_1$, $z(t) = y(kt)$ is defined in the interval $]b, 1[ \,$ where $(0 \leq b = (1/k_1) a < 1)$, since for $k \geq k_1$, the transformation $t \to kt$ maps the interval $]b, 1[ \,$ onto the interval $]kb, k[ \subset ]a, \infty[ .$

II. The case $m + r < 0$. Let the equation (1.1) be defined in some interval $]0, a[ \,$ where $0 < k_1 < a \leq \infty$ so that $k_1$ is still an inner point of the domain of the equation (1.1). Put $a = (l/k_1)$ ($l > 1$). For $0 < k \leq k_1$, the transformation $t \to kt$ maps the interval $]1, b[ \,$ onto the interval $]k, kb[ \subset ]0, a[ \,$ and thus the function $z(t) = y(kt)$ is defined in the interval $[1, b[ .$

In the case I the function $z_i(t), i = 1, 2, 3, \ldots \,$ satisfies the differential equation (1.2) with the parameter $\lambda = \lambda_i$ in the interval $]b, 1[ \,$ and thus for $\beta \in ]b, 1[ \,$ it holds

\begin{equation}
[\Theta(z_i'z_j - z_i'z_j)]_{\beta} + (\lambda_i - \lambda_j) \int_{\beta}^{1} A z_i z_j \, dt = 0, \quad i, j = 1, 2, 3, \ldots
\end{equation}

Hence it follows that the sequence of functions $z_1, z_2, z_3, \ldots \,$ is in the interval $]b, 1[ \,$ orthogonal with the weight $A(t)$ iff for any $i \neq j$

\begin{equation}
\lim_{\beta \to b^+} [\Theta(\beta) [z_i'(\beta) z_j(\beta) - z_i(\beta) z_j'(\beta)] = 0 .
\end{equation}

In the case II the function $z_i(t), i = 1, 2, 3, \ldots \,$ satisfies the differential equation (1.2) with the parameter $\lambda = \lambda_i$ in the interval $[1, b[ \,$ and thus for $\beta \in [1, b[ \,$ it holds

\begin{equation}
[\Theta(z_i'z_j - z_i'z_j)]_{\beta} + (\lambda_i - \lambda_j) \int_{1}^{\beta} A z_i z_j \, dt = 0, \quad i = 1, 2, 3, \ldots
\end{equation}

Hence it follows that the sequence of functions $z_1, z_2, z_3, \ldots \,$ is in the interval $[1, b[ \,$ orthogonal with the weight $A(t)$ iff for any $i \neq j$

\begin{equation}
\lim_{\beta \to b^-} [\Theta(\beta) [z_i'(\beta) z_j(\beta) - z_i(\beta) z_j'(\beta)] = 0 .
\end{equation}

Example. For the Bessel equation

\begin{equation}
(ty')' - \frac{n^2}{t} y = -ty, \quad n \geq 0 \quad \text{fixed}
\end{equation}

in the interval $]0, \infty[ \,$ we have $\Theta(t) = t, B(t) = n^2/t, A(t) = t, m = 1, r = 1.$ In the interval $]0, \infty[ \,$ the solution $J_n(t)$ has infinitely many roots $k_i, i = 1, 2, 3, \ldots$ increasing to $\infty$ so that the case I occurs. The functions $J_n(k_i t), i = 1, 2, 3, \ldots \,$ form in the interval $]0, 1[ \,$ an orthogonal sequence with the weights $t$ iff for $i \neq j$

\begin{equation}
\lim_{t \to 0^+} \{ t [J_n'(k_i t) J_n(k_i t) - k_j J_n(k_i t) J_n'(k_j t)] \} = 0 .
\end{equation}
According to the formula \( J'_n(x) = -J_{n+1}(x) + (n|x) J_n(x) \), the expression \( p(t) \) following the limit symbol in (1.8) is reduced to \( p(t) = p_1(t) - p_2(t) \) where \( p_1(t) = -tk_j J_{n+1}(k_j t) J_n(k_j t) \) and \( p_2(t) = tk_j J_{n+1}(k_j t) J_n(k_j t) \).

Consider that the following two rules hold for the asymptotic equality \( \sim \):

1° \( a_i \sim b_i, \ i = 1, 2 \Rightarrow a_1 a_2 \sim b_1 b_2 \),

2° \( a \sim b, \ b \to 0 \Rightarrow a \to 0 \).

From the formula \( J_n(x) \sim x^n/(2^n \Gamma(1 + n)) \) for \( x \to 0 \) we have then

\[
P_1(t) \sim \frac{k_j}{2^{2n+1} \Gamma(2 + n) \Gamma(1 + n)} t^{n+1},
\]

\[
P_2(t) \sim \frac{k_i}{2^{2n+1} \Gamma(2 + n) \Gamma(1 + n)} t^{n+1}
\]

so that \( p_i(t) \to 0, \ i = 1, 2 \) holds and thus \( p(t) \to 0 \) for \( t \to 0 \) iff \( n + 1 > 0 \).

2. Let the linear differential operator of the \( n \)-th order in the complex domain

(2.1) \( s_L y(t) = \sum_{i=1}^{n} a_i(t) y^{(i)}(t) \)

have the following property: After the linear substitution \( t \to kt, \ k \in \mathbb{C} \), it fulfils for a suitable \( m \in \mathbb{C} \) the relation

(2.2) \( s_L y(kt) = k^m s_L y(kt) \).

Let the differential equation

(2.3) \( s_L y(t) = \lambda y(t) \)

have a solution \( y(t) \) for a constant \( \lambda \in \mathbb{C} \). Then the function \( y(kt) \) satisfies the equation

(2.4) \( s_L y(kt) = \lambda k^m y(kt) \).

Form an equation of the \( 2n \)-th order with the operator \( s_L^2 = s_L s_L \) and constants \( p, q \in \mathbb{C} \)

(2.5) \( s_L^2 y(t) + 2p s_L y(t) + q y(t) = 0 \).

Look for its solution in the form \( y(kt) \) where \( y(t) \) is a solution of (2.3) and \( k \) is a suitable constant. We get "a charakteristic" equation for the unknown \( k \)

(2.6) \( (\lambda k^m)^2 + 2p(\lambda k^m) + q = 0 \).

For any \( k \) fulfilling (2.6) and for any \( y(t) \) fulfilling (2.3) the function \( y(kt) \) then fulfils (2.5).
In case of \( p^2 = q \) the equation (2.6) has a double root \( k^m = -p/\lambda \). The corresponding differential equation

\[(2.7) \quad (\lambda + p)^2 \ y(t) = 0 \]

has a solution \( y(t) \) iff \( y(t) \) satisfies the equation

\[(2.8) \quad (\lambda + p) \ y(t) = z(t) \]

where \( z(t) \) is a suitable solution of the equation

\[(2.9) \quad (\lambda + p) \ z(t) = 0 . \]

The last mentioned assertions hold generally for any operator \( A : M \to M \) on any set \( M \): for \( b \in M \) the equation \( A^2 y = b \) is equivalent to the equations \( Ay = z, Az = b \).

3. For arbitrary \( n \in \mathbb{C} \) put, in the complex domain,

\[(3.1) \quad ^nE \ y(t) = y''(t) + \frac{1}{t} \ y'(t) - \frac{n^2}{t^2} \ y(t) . \]

Then the Bessel equation of the index \( n \) may be written in the form

\[(3.2) \quad ^nE \ y(t) = -y(t) \]

or

\[(3.3) \quad (^nE + 1) \ y(t) = 0 . \]

For an arbitrary \( k \in \mathbb{C} \) and for an arbitrary solution \( y(t) \) of the equation (3.3) the function \( z(t) = y(kt) \) is a solution of the equation

\[(3.4) \quad (^nE + k^2) \ z(t) = 0 \]

as the operator \( ^nE \ y(t) \) has the property (2.2) for \( m = 2 \). From this property it also follows that, if \( y(t) \) is a solution of the equation

\[(3.5) \quad (^nE + 1) \ y(t) = f(t) \]

where \( f \) is an arbitrary continuous function, then for arbitrary \( k \in \mathbb{C} \) the function \( z(t) = y(kt) \) is a solution of the equation

\[(3.6) \quad (^nE + k^2) \ z(t) = k^2 f(kt) . \]

Consider the iterated equation \((p, q \in \mathbb{C})\)

\[(3.7) \quad (^nE^2 + 2p^nE + q) \ y(t) = 0 . \]
In case of \( y(t) \) being a solution of the Bessel equation (3.3), \( y(kt) \) is a solution of the equation (3.7) iff

\[
k_{1,2}^2 = p \pm \sqrt{(p^2 - q)}.
\]

Combinations of the four values \( \pm k_1, \pm k_2 \in \mathbb{C} \) and of the two linearly independent solutions \( J_n(t), Y_n(t) \) of the equation (3.3) yield eight solutions of the equation (3.7). Since it holds for \( m \in \mathbb{Z}, n \in \mathbb{C} \) (\( \mathbb{Z} \) is the set of all integers)

\[
J_n(te^{im\pi}) = e^{inm} J_n(t),
\]

\[
Y_n(te^{im\pi}) = e^{-imn} Y_n(t) + 2i \frac{\sin m\pi n}{\sin n\pi} \cos n\pi J_n(t),
\]

we can cancel the four solutions containing the arguments \(-k_1t, -k_2t\), because they are linear combinations of the others. The remaining solutions \( J_n(k_1t), J_n(k_2t), Y_n(k_1t), Y_n(k_2t) \) are linearly independent iff \( k_1 \neq k_2 \).

**Proof.** Take \( a J_n(k_1t) + b J_n(k_2t) + c Y_n(k_1t) + d Y_n(k_2t) = 0 \). Put \( y(t) = a J_n(k_1t) + c Y_n(k_1t) = -b J_n(k_2t) - d Y_n(k_2t) \). Then \( y(t) \) is a solution of the equation (3.4) for \( k = k_1 \) and \( k = k_2 \) so that \( k_1^2 y(t) = k_2^2 y(t) \). Hence in case of \( k_1 \neq k_2 \) we have \( y(t) = 0 \) and then \( a = c = 0, b = d = 0 \), Q.E.D.

In case of \( k_1 = k_2 = k \), i.e. by \( p^2 = q \), we get only two linearly independent solutions \( J_n(kt), Y_n(kt) \) of the equation (3.7), which is now of the form

\[
(\xi^2 + p)^2 y(t) = 0.
\]

Since \( k_1^2 = p \) it is

\[
(\xi^2 + k_2^2)^2 y(t) = 0.
\]

Let \( a, b \in \mathbb{C} \) be arbitrary fixed constants. Then the function \( Z_n(t) = a J_n(t) + b Y_n(t) \) is called a „general“ cylindrical function of the index \( n \). Since it is a fixed linear combination of the functions \( J_n(t), Y_n(t) \) with coefficients independent of the index \( n \), the same recurrent relations hold for \( Z_n(t) \) as for \( J_n(t) \) and \( Y_n(t) \), e.g.

\[
t Z_n'(t) - n Z_n(t) = -t Z_{n+1}(t).
\]

The Bessel equation of the index \( n \) in the self-adjoint form is

\[
(ty')' + \left( t - \frac{n^2}{t} \right) y = 0.
\]

Put \( y(t) = t Z_{n+1}(t), [2] \). From the relation (3.13) we get

\[
y(t) = n Z_n(t) - t Z'_n(t).
\]
Differentiating, multiplying by $t$ and differentiating once more we get

\[ (ty')' = n(tZ_n') + (t^2 - n^2) Z_n' + 2tZ_n , \]

and once more by (3.13) we find

\[ (ty')' + \left( t - \frac{n^2}{t} \right) y = 2tZ_n \]

or

\[ (\tau E + 1) y(t) = 2Z_n(t) . \]

From the considerations concerning (2.8) it appears that $y(t) = t \, Z_{n+1}(t)$ is a solution of the equation

\[ (\tau E + 1)^2 y(t) = 0 . \]

According to (3.5), (3.6) it follows from (3.18) that the function $z(t) = y(kt) = kt \, Z_{n+1}(kt)$ is a solution of the equation

\[ (\tau E + k^2) z(t) = 2k^2 \, Z_n(kt) \]

so that $z(t) = y(kt) = kt \, Z_{n+1}(kt)$ satisfies the equation (3.12). So we find that the functions $t \, J_{n+1}(kt)$, $t \, Y_{n+1}(kt)$ are again solutions of the equation (3.12). At the same time the solutions $J_n(kt)$, $Y_n(kt)$, $t \, J_{n+1}(kt)$, $t \, Y_{n+1}(kt)$ of the equation (3.12) are linearly independent.

**Proof.** Consider a linear relation $a \, J_n(kt) + b \, Y_n(kt) + ct \, J_{n+1}(kt) + dt \, Y_{n+1}(kt) = 0$. Put $Z_n(t) = -(c/k) \, J_n(t) - (d/k) \, Y_n(t)$. Then the function $kt \, Z_{n+1}(kt) = -ct \, J_{n+1}(kt) - dt \, Y_{n+1}(kt) = a \, J_n(t) + b \, Y_n(t)$ is a solution of both equations (3.20) and (3.12). Hence it follows that $Z_n = 0$ so that $c = d = 0$ as well as $a = b = 0$; Q.E.D.

**References**

[1] **B. G. Korenjev:** Some elasticity and heat conduction problems solvable in Bessel functions (Russian), Moscow 1960.


Souhrn

NĚKTERÉ VLASTNOSTI LINEÁRNÍ HOMOGENNÍ TRANSFORMACE NEZÁVISLE PROMĚNNÉ V OBYČEJNÝCH DIFERENCIÁLNÍCH LINEÁRNÍCH ROVNICÍCH

Erich Barvínek

Odst. 1 obsahuje obecnou lineární diferenciální rovnici 2. řádu (1.1), jejíž řešení $y(t)$ vytváří orthogonální posloupnost $y(k_t)$, kde $k_i$ je vhodně uspořádaná posloupnost kladných kořenů řešení $y(i)$. Jde v podstatě o „Eulerovské“ rovnice.


\[
\left( \frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt} \right)^2 w - 2b_0 \left( \frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt} \right) w + w = 0
\]

v tom smyslu, že je nalezeno obecné řešení rovnice (3.7).

Pozoruhodná věta: je-li $Z_n(t) = a J_n(t) + b Y_n(t)$ libovolné řešení (3.3), $k \in \mathbb{C}$ libovolně, pak

1° $Z_n(kt)$ je řešení (3.4),

2° $kt Z_{n+1}(kt)$ je řešení (3.20) a tudiž (3.12),

je rozšířením úvah [2] o nalezení řešení $t Z_{n+1}(t)$ rovnice (3.12) a skýtá důkaz lineární nezávislosti jejich řešení $J_n(kt)$, $Y_n(kt)$, $t J_{n+1}(kt)$, $t Y_{n+1}(kt)$.

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