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ON A FUNCTIONAL EQUATION

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Let us consider a region $\mathcal{D}$ in a Banach space and a system S of strategies, each strategy being a transformation of $\mathcal{D}$ into $\mathcal{D}$.

We shall make an n-step procedure; we depart from an initial point x, choose n strategies $q_1, \ldots, q_n$ consecutively and pass through n points $x_1 = q_1 x, \ldots, x_n = q_n x_{n-1}$. This choice yields the result

$$R_n(x; q_1, \ldots, q_n) .$$

Let us write

$$\Sigma_n(x) = \sup_{(q_1, \ldots, q_n)} R_n(x; q_1, \ldots, q_n) . \tag{1}$$

We have

$$\Sigma_n(x) = \sup_{q_1} \left(\sup_{(q_2, \ldots, q_n)} R_n(x; q_1, q_2, \ldots, q_n) \right) .$$

We shall suppose (similarly to the principle of optimum [1]) that the final result R_n depends in some manner on the initial point x, the first chosen point x_1 and the result R_{n-1} of subsequent choice $x_2, \ldots, x_n$ the first choice x_1 made.

Under this assumption we can write

$$R_n(x; q_1, \ldots, q_r) = F(x, q_1, R_{n-1}(x_1; q_2, \ldots, q_n))$$

or rather more generally

$$R_n(x; q_1, \ldots, q_n) = F(x, q_1, R_{r-1}(f(x, q_1); q_2, \ldots, q_n))$$

where $F(f)$ is a function of three (two) variables.

Supposing that F is an increasing function of the last aigument we have

$$\Sigma_n(x) = \sup_{q_1} \sup_{(q_2, \ldots, q_n)} F(x\ q_1, R_{n-1}(f(x, q_1); q_2, \ldots, q_n)) =$$
$$= \sup_{q_1} F(x, q_1, \Sigma_{n-1}(f(x, q_1))) .$$

$\Sigma(x) = \lim \Sigma_n(x)$ satisfies the following equation

$$\Sigma(x) = \sup_{q \in S} F(x, q, \sum(f(x, q))) \tag{2}$$

(we suppose that $\Sigma(x)$ exists and the interchanging of limits on the right-hand side is allowed).

We shall show that

the solution of the equation (2) *exists under the following conditions*:

(i) $\mathscr{D}$ is a neighbourhood of the origin (in a Banach space),

(ii) F satisfies Lipschitz condition in the third argument

$$|F(x, q, z) - F(x, q, z')| \leqq a|z - z'|,$$

(iii) $f(x, q) \in \mathscr{D}$ for all $(x, q) \in \mathscr{D} \times S$ and $\|f(x, q)\| \leqq c\|x\|$,

(iv) $\varrho = ac < 1$,

(v) $|F(x, q, 0)| \leqq B\|x\|$

and this solution is unique in the class of bounded functionals.

Proof. We choose a functional σ_0 with

$$|\sigma_0(x)| \leqq A\|x\| \tag{3}$$

and proceed by induction

$$\sigma_{n+1}(x) = \sup_q F(x, q, \sigma_n(f(x, q))). \tag{4}$$

It is

$$|\sigma_{n+1}(x) - \sigma_n(x)| = |\sup F(\ldots, \sigma_n) - \sup F(\ldots, \sigma_{n-1})| \leqq$$
$$\leqq \sup |F(\ldots, \sigma_n) - F(\ldots, \sigma_{n-1})| \leqq a \sup |\sigma_n(f(x, q)) - \sigma_{n-1}(f(x, q))|.$$

We make the assumption

$$|\sigma_i(x) - \sigma_{i-1}(x)| \leqq b\varrho^i\|x\|, \quad 2 \leqq i \leqq n \tag{5_i}$$

and we have

$$|\sigma_{n+1}(x) - \sigma_n(x)| \leqq ab\varrho^n c\|x\| = b\varrho^{n+1}\|x\|. \tag{5_{n+1}}$$

It remains to evaluate

$$|\sigma_1(x) - \sigma_0(x)| = |\sup_q F(x, q, \sigma_0(f(x, q))) - \sigma_0(x)|. \tag{6}$$

The right-hand side of (6) is less than

$$|\sup (F(x, q, \sigma_0) - F(x, q, 0))| + |\sup F(x, \varrho, 0)| + |\sigma_0(x)|.$$

The first term is (following (ii), (iii) and (3))

$$\leqq a|\sigma_0(f(x, q))| \leqq aAc\|x\| ,$$

hence

$$|\sigma_1(x) - \sigma_0(x)| \leqq aAc\|x\| + B\|x\| + A\|x\| = [(ac + 1) A + B] \|x\|$$

and this can be made less than $b\varrho = bac$ by the choice

$$b \geqq [(ac + 1) A + B] (ac)^{-1} .$$

The sum $\sigma(x) = \sigma_0(x) + \sum_{n=0}^{\infty} (\sigma_{n+1}(x) - \sigma_n(x))$ exists by (iv) and we have $\sigma(x) = \lim \sigma_n(x)$ and

$$(7) \qquad |\sigma(x)| \leqq c\|x\| .$$

Now we shall show that $\sigma(x)$ is the unique solution of the equation (2) in the class of bounded functionals (i.e. satisfying (7)).

Because we do not suppose more than (iii) and (v) about the function F we proceed in the following manner:

We have

$$|\sup_q F(x, q, \sigma(f(x, q))) - \sigma_n(x)| \leqq a \sup_q |\sigma(f(x, q)) - \sigma_{n-1}(f(x, q))| .$$

Now $|\sigma(x) - \sigma_{n-1}(x)| \leqq \sum_{j=n}^{\infty} |\sigma_j(x) - \sigma_{j-1}(x)|$ and by (5_j) we have

$$|\sup_q F(x, q, \sigma(f(x, q))) - \sigma_n(x)| \to 0$$

and so σ is a solution.

Let σ be a fixed solution of equation (2) and let us consider another sequence $\sigma'_0, \sigma'_1, \ldots$ where σ'_0 is a bounded functional and $\sigma'_{i+1} = \sup F(x, q, \sigma'_i(f(x, q)))$. Both σ, σ'_0 are bounded and hence

$$|\sigma(x) - \sigma'_0(x)| \leqq \beta\|x\| .$$

If we suppose

$$|\sigma(x) - \sigma'_n(x)| \leqq \beta\varrho^n\|x\|$$

then we have

$$|\sigma(x) - \sigma'_{n+1}(x)| = |\sup F(x, q, \sigma(f(x, q))) - \sup F(x, q, \sigma'_n(f(x, q)))| \leqq$$
$$\leqq a \sup |\sigma(f(x, q)) - \sigma'_n(f(x, q))| \leqq \beta ac\varrho^n\|x\| .$$

Hence by induction $\sigma'_n(x) \to \sigma(x)$.

When we have another solution σ' then we choose $\sigma'_0 = \sigma'$ and it is $\sigma'_n = \sigma'$ for all n and therefore $\sigma' = \sigma$.

Reference

[1] *R. Bellman:* Dynamic Programming, Priceton Univ. Press 1957, Princeton, N.J.

Souhrn

O JEDNÉ FUNKCIONÁLNÍ ROVNICI

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Dokazuje se, že rovnice

$$\sigma(x) = \sup_q F(x, q, \sigma(f(x, q))) \tag{2}$$

za podmínek (i)–(v) má právě jedno řešení. Rovnice (2) řeší úlohu o optimálním výsledku při neomezeně vzrůstajícím počtu rozhodovacích kroků v Banachově prostoru.

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