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ON GENERALIZED LOCALIZABILITY

Václav Alda

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The localizability of a physical system $S$ in a space $X$ is defined as a homomorphism $h$ of the lattice $B(X)$ of the Borel sets in the space $X$ into the lattice $\mathcal{E}$ of yes — no experiments which can be performed on $S$. In each state $s \in \mathcal{S}$ on $\mathcal{E}$ and $A \in B(X)$, $s(h(A))$ is the probability of finding $S$ in $A$. In [1] Jauch and Piron introduced generalized localizability; the condition that $h: B(X) \to \mathcal{S}$ is a homomorphism is weakened. They demand

1. $h(\emptyset) = 0$, $h(X) = 1$,
2. $A_1 \cap A_2 = \emptyset \Rightarrow h(A_1) \perp h(A_2)$,
3. $h(A_1 \cap A_2) = h(A_1) \land h(A_2)$,

but they do not demand $h(A_1 \cup A_2) = h(A_1) \lor h(A_2)$ for $A_1 \cap A_2 = \emptyset$. In their paper [1] $\mathcal{S}$ is the lattice of projections in a Hilbert space $\mathcal{H}$.

A transformation $h: B(X) \to \mathcal{S}$ satisfying (1) – (3) can be constructed in a simple manner. We take a homomorphism $h': B(X) \to \mathcal{S}$, a projection $P$ non-commuting with all $h'(A)$, $A \in B(X)$, and then

$$h : h(A) = P \wedge h'(A)$$

satisfies (1) – (3) in $P\mathcal{H}$.

This is the situation which is described in a theorem due to Neumark [2]: For every positive operator valued measure $T$ in a Hilbert space $\mathcal{H}$ one can construct an extension $\mathcal{S}' \supset \mathcal{S}$ and a projection valued measure $T'$ so that

$$T(A) = P T'(A) P$$

where $P$ is the projection $\mathcal{S}' \to \mathcal{S}$. The converse that $T$ constructed in that manner is a POV-measure is also true.

Now Jauch and Piron made the following conjecture: For every generalized spectral measure $h: B(X) \to \mathcal{S}(\mathcal{H})$ which satisfies (1) – (3) there exists an extension
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§ and a spectral measure \( \mathfrak{h}': B(K) \rightarrow \mathfrak{S}' \) so that \( \mathfrak{h}(A) = P \wedge \mathfrak{h}'(A) \) where \( P \) is the projection \( \mathfrak{S}' \rightarrow \mathfrak{S} \).

By the theorem of Neumark there must be a POV-measure on \( B(X) \).

The existence of this measure can be proved without the detour over \( h \) satisfying (1)–(3). It suffices to define the localizability of a system \( S \) in a space \( X \) in the following manner:

\( (\text{A1}) \) for every \( A \in B(X) \) and every state \( s \in \mathcal{S} \) of the system \( S \) there is a probability \( p(A, s) \) for \( S \) in \( A \). This function is additive in \( A \) and linear in \( s \) so that \( A_1 \cap A_2 = \emptyset \Rightarrow p(A_1 \cup A_2, \cdot) = p(A_1, \cdot) + p(A_2, \cdot) \) and \( s_1, s_2 \in \mathcal{S}, \; \alpha, \beta \geq 0, \; \alpha + \beta = 1 \Rightarrow p(\cdot, \alpha s_1 + \beta s_2) = \alpha p(\cdot, s_1) + \beta p(\cdot, s_2) \).

In order that this probability may be experimentally determined, we must suppose

\( (\text{A2}) \) for every \( A \) there is an observable \( P(A) \) with the mean value \( p(A, s) \) in every state \( s \).

The definition requires that the sum of the observables \( P(A_1), P(A_2) \) should exist, at least for \( A_1 \cap A_2 = \emptyset \) (cf. [4]). In the sequel we shall deal with the case that \( \mathcal{S} \) is a Boolean algebra or the lattice of projections in a Hilbert space \( \mathfrak{S} \), and so this condition will be satisfied.

In the second case the observables \( P(A) \) are operators on \( \mathfrak{S} \) and, in order that the mean value be from the interval \( \langle 0, 1 \rangle \) for every state, it is necessary and sufficient that \( P(A) \) be a positive operator and \( P(A) \leq I \) (I — the unit projection). The condition \( P(X) = I \) is equivalent to \( p(X, s) = 1 \) for every state \( s \).

In the classical case, where \( \mathcal{S} \) is a Boolean algebra, we can consider the sets in a compact space \( T \) in place of \( \mathcal{S} \) (Theorem of Stone). \( P(A) \) are then measurable functions on \( T \) and states are measures on \( T \). The necessary and sufficient condition that the mean value lie in \( \langle 0, 1 \rangle \) is that the values of \( P \) lie in \( \langle 0, 1 \rangle \). This follows from the fact that for every event \( x \in \mathcal{S} \) it is possible to find a state \( s \in \mathcal{S} \) that \( s(x) = 1 \) provided that the structure \( (\mathcal{S}, \mathcal{F}) \) is strongly-order—determining [3] (cf. [4]). In this case, therefore, the localizability is a system of functions \( P(A), A \in B(X) \), with values in \( \langle 0, 1 \rangle \) and \( P(A_1 \cup A_2) = P(A_1) + P(A_2) \) for \( A_1 \cap A_2 = \emptyset \). \( P(X) = 1 \) is always equivalent to \( S \in X \).

Now the following statement is valid (this statement is an analogue of the statement from [2]):

It is possible to find for \( \mathcal{S} \) an extension \( \mathcal{S}' \) and a system of states \( \mathcal{S}' \) such that there is an isomorphism \( h: B(X) \rightarrow \mathcal{S}' \) and for every state \( s \in \mathcal{S} \) there is a state \( s' \in \mathcal{S}' \) so that the probability \( S \in A \) in the state \( s \) is equal to the probability of \( h(A) \) in the state \( s' \).

Proof. We shall consider the representation of the algebra \( \mathcal{S} \) in the space \( T \). In the product \( X \times T \) we have sets \( A \times M, A \in B(X), M \in \mathcal{S} \). Now we define the measure \( s' \) on \( X \times T \) for \( s \in \mathcal{S} \)

\[ s'(A \times M) = \int_M P(A) \, ds \]
s' is a measure because \( P \) is additive in \( A \). We can extend \( s' \) on the algebra \( \mathcal{E}' \) generated by the sets of the form \( A \times M \). In this algebra \( \mathcal{E}' \) the sets \( X \times M \) form an subalgebra that is isomorphic to \( \mathcal{E} \). For every \( A \) we define \( h(A) = A \times T \) and \( h \) is an isomorphism. Finally it is

\[
s'(h(A)) = \int_T P(A) \, ds = p(A, s).
\]

This completes the proof.

The part about the states is valid for \([2]\), too:

*It is possible to extend every state \( s \) in \( \mathcal{S} \) to a state \( s' \) in \( \mathcal{S}' \) so that we have \( s'(h(A)) = s(P(A)) \) (in \([2]\) there is \( F \) in place of \( h \) and \( B \) in place of \( P \)).*

**Proof.** When the state \( s \) is irreducible then there is a vector \( \phi \in \mathcal{S} \) that defines this state. \( h(A) \) is formed by all sums \( \Sigma_k A_k \cdot x_k \) (cf. \([2]\)) where \( A_k \in A \). It follows from this that \( (X - A) \phi \) is orthogonal to \( h(A) \) and from the equality \( \phi = X\phi = A\phi + (X - A) \phi \) it follows that \( A\phi \) is the projection of \( \phi \) into \( h(A) \). From the definition of the scalar product in \( \mathcal{S}' \) the equation

\[
(h(A) \phi, \phi)_{\mathcal{S}'} = (P(A) \phi, \phi)_{\mathcal{S}}
\]

follows. The validity for composed states is a consequence of linearity.

**References**


**O ZOBEČNĚNÉ LOKALIZOVATELNOSTI**

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Ukazuje se, že definice zobecněné lokalizovatelnosti podle Jaucha a Pirona může být přímo zdůvodněna. Je dokázána věta pro klasické systémy, která je analogem Neumarkovy věty o representaci POV měr.

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