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ON GENERALIZED LOCALIZABILITY

VÁCLAV ALDA

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The localizability of a physical system S in a space X is defined as a homomorphism h of the lattice $B(X)$ of the Borel sets in the space X into the lattice $\mathscr{E}$ of yes – no experiments which can be performed on S. In each state $s \in \mathscr{S}$ on $\mathscr{E}$ and $\Delta \in B(X)$, $s(h(\Delta))$ is the probability of finding S in Δ. In [1] Jauch and Piron introduced generalized localizability; the condition that h: $B(X) \to \mathscr{E}$ is a homomorphism is weakened. They demand

$$h(\emptyset) = 0\,, \quad h(X) = I\,, \tag{1}$$

$$\Delta_1 \cap \Delta_2 = \emptyset \Rightarrow h(\Delta_1) \perp h(\Delta_2)\,, \tag{2}$$

$$h(\Delta_1 \cap \Delta_2) = h(\Delta_1) \wedge h(\Delta_2)\,, \tag{3}$$

but they do not demand $h(\Delta_1 \cup \Delta_2) = h(\Delta_1) \vee h(\Delta_2)$ for $\Delta_1 \cap \Delta_2 = \emptyset$. In their paper [1] $\mathscr{E}$ is the lattice of projections in a Hilbert space $\mathfrak{H}$.

A transformation h: $B(X) \to \mathscr{E}$ satisfying (1)–(3) can be constructed in a simple manner. We take a homomorphism h': $B(X) \to \mathscr{E}$, a projection P non-commuting with all $h'(\Delta)$, $\Delta \in B(X)$, and then

$$h : h(\Delta) = P \wedge h'(\Delta)$$

satisfies (1)–(3) in $P\mathfrak{H}$.

This is the situation which is described in a theorem due to Neumark [2]: For every positive operator valued measure T in a Hilbert space $\mathfrak{H}$ one can construct an extension $\mathfrak{H}' \supset \mathfrak{H}$ and a projection valued measure T' so that

$$T(\Delta) = P\, T'(\Delta)\, P$$

where P is the projection $\mathfrak{H}' \to \mathfrak{H}$. The converse that T constructed in that manner is a POV-measure is also true.

Now Jauch and Piron made the following conjecture: For every generalized spectral measure h: $B(X) \to \mathscr{E}(\mathfrak{H})$ which satisfies (1)–(3) there exists an extension

$\mathfrak{H}' \supset \mathfrak{H}$ and a spectral measure $h'\colon B(X) \to \mathscr{E}(\mathfrak{H}')$ so that $h(\Delta) = P \wedge h'(\Delta)$ where P is the projection $\mathfrak{H}' \to \mathfrak{H}$.

By the theorem of Neumark there must be a POV-measure on $B(X)$.

The existence of this measure can be proved without the detour over h satisfying (1)–(3). It suffices to define the localizability of a system S in a space X in the following manner:

(A1) for every $\Delta \in B(X)$ and every state $s \in \mathscr{S}$ of the system S there is a probability $p(\Delta, s)$ for S in Δ. This function is additive in Δ and linear in s so that $\Delta_1 \cap \Delta_2 = \emptyset \Rightarrow \Rightarrow p(\Delta_1 \cup \Delta_2, .) = p(\Delta_1, .) + p(\Delta_2, .)$ and $s_1, s_2 \in \mathscr{S}$, $\alpha, \beta \geqq 0$, $\alpha + \beta = 1 \Rightarrow \Rightarrow p(., \alpha s_1 + \beta s_2) = \alpha\, p(., s_1) + \beta\, p(., s_2)$.

In order that this probability may be experimentally determined, we must suppose

(A2) for every Δ there is an observable $P(\Delta)$ with the mean value $p(\Delta, s)$ in every state s.

The definition requires that the sum of the observables $P(\Delta_1)$, $P(\Delta_2)$ should exist, at least for $\Delta_1 \cap \Delta_2 = \emptyset$ (cf. [4]). In the sequel we shall deal with the case that $\mathscr{E}$ is a Boolean algebra or the lattice of projections in a Hilbert space $\mathfrak{H}$, and so this condition will be satisfied.

In the second case the observables $P(\Delta)$ are operators on $\mathfrak{H}$ and, in order that the mean value be from the interval $\langle 0, 1 \rangle$ for every state, it is necessary and sufficient that $P(\Delta)$ be a positive operator and $P(\Delta) \leqq I$ (I – the unit projection). The condition $P(X) = I$ is equivalent to $p(X, s) = 1$ for every state s.

In the classical case, where $\mathscr{E}$ is a Boolean algebra, we can consider the sets in a compact space T in place of $\mathscr{E}$ (Theorem of Stone). $P(\Delta)$ are then measurable functions on T and states are measures on T. The necessary and sufficient condition that the mean value lie in $\langle 0, 1 \rangle$ is that the values of P lie in $\langle 0, 1 \rangle$. This follows from the fact that for every event $x \in \mathscr{E}$ it is possible to find a state $s \in \mathscr{S}$ that $s(x) = 1$ provided that the structure $(\mathscr{E}, \mathscr{S})$ is strongly-order – determining [3] (cf. [4]). In this case, therefore, the localizability is a system of functions $P(\Delta)$, $\Delta \in B(X)$, with values in $\langle 0, 1 \rangle$ and $P(\Delta_1 \cup \Delta_2) = P(\Delta_1) + P(\Delta_2)$ for $\Delta_1 \cap \Delta_2 = \emptyset$. $P(X) = 1$ is always equivalent to S in X.

Now the following statement is valid (this statement is an analogue of the statement from [2]):

It is possible to find for $\mathscr{E}$ an extension $\mathscr{E}'$ and a system of states $\mathscr{S}'$ such that there is an isomorphism $h\colon B(X) \to \mathscr{E}'$ and for every state $s \in \mathscr{S}$ there is a state $s' \in \mathscr{S}'$ so that the probability $S \in \Delta$ in the state s is equal to the probability of $h(\Delta)$ in the state s'.

Proof. We shall consider the representation of the algebra $\mathscr{E}$ in the space T. In the product $X \times T$ we have sets $\Delta \times M$, $\Delta \in B(X)$, $M \in \mathscr{E}$. Now we define the measure s' on $X \times T$ for $s \in \mathscr{S}$

$$s'(\Delta \times M) = \int_M P(\Delta)\, \mathrm{d}s\,.$$

s' is a measure because P is additive in Δ. We can extend s' on the algebra $\mathcal{E}'$ generated by the sets of the form $\Delta \times M$. In this algebra $\mathcal{E}'$ the sets $X \times M$ form an subalgebra that is isomorphic to $\mathcal{E}$. For every Δ we define $h(\Delta) = \Delta \times T$ and h is an isomorphism. Finally it is

$$s'(h(\Delta)) = \int_T P(\Delta)\, \mathrm{d}s = p(\Delta, s)\,.$$

This completes the proof.

The part about the states is valid for [2], too:

It is possible to extend every state s in $\mathfrak{H}$ to a state s' in $\mathfrak{H}'$ so that we have $s'(h(\Delta)) = s(P(\Delta))$ (in [2] there is F in place of h and B in place of P).

Proof. When the state s is irreducible then there is a vector $\varphi \in \mathfrak{H}$ that defines this state. $h(\Delta)$ is formed by all sums $\Sigma_k \Delta_k \,.\, x_k$ (cf. [2]) where $\Delta_k \subset \Delta$. It follows from this that $(X - \Delta)\,\varphi$ is orthogonal to $h(\Delta)$ and from the equality $\varphi = X\varphi = \Delta\varphi + (X - \Delta)\,\varphi$ it follows that $\Delta\varphi$ is the projection of φ into $h(\Delta)$. From the definition of the scalar product in $\mathfrak{H}'$ the equation

$$(h(\Delta)\,\varphi, \varphi)_{\mathfrak{H}'} = (P(\Delta)\,\varphi, \varphi)_{\mathfrak{H}}$$

follows. The validity for composed states is a consequence of linearity.

References

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[3] *J. C. T. Pool:* Baer*-Semigroups and the Logic of Quantum Mechanics, Comm. Math. Physics 9 (1968), 118.

[4] *S. P. Gudder:* Uniqueness and existence properties of bounded observables, Pacific J. Math. 19 (1966), 81.

Souhrn

O ZOBECNĚNÉ LOKALIZOVATELNOSTI

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Ukazuje se, že definice zobecněné lokalizovatelnosti podle Jaucha a Pirona může být přímo zdůvodněna. Je dokázána věta pro klasické systémy, která je analogem Neumarkovy věty o representaci POV měr.

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