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ALGORITMY

32. SOMMERFELD COX

COMPUTATION OF SOMMERFELD'S ATTENUATION FUNCTION

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This procedure computes the complex-valued Sommerfeld attenuation function, $G(p)$, which appears within the theory of propagation of electromagnetic waves [11]:

$$G(p) = 1 + \mathrm{i}\sqrt{(\pi p)}\, e^{-p} \operatorname{erfc}(-\mathrm{i}\sqrt{p})$$

where

$$\operatorname{erfc}(-\mathrm{i}\sqrt{p}) = \frac{2}{\sqrt{\pi}} \int_{-\mathrm{i}\sqrt{p}}^{\infty} e^{-t^2}\, \mathrm{d}t\,,$$

provided that $0 \leqq \arg(p) \leqq \pi/2$. This function has been tabulated [7].

By means of the function $w(z)$ [6], defined by

$$w(z) = e^{-z^2}\left(1 + \frac{2\mathrm{i}}{\sqrt{\pi}} \int_0^z e^{t^2}\, \mathrm{d}t\right) = \mathrm{e}^{-z^2} \operatorname{erfc}(-\mathrm{i}z)$$

the function $G(p)$ can be expressed as

$$G(p) = 1 + \mathrm{i}\sqrt{(\pi p)}\, w(\sqrt{p})\,.$$

The function $w(z)$ can be approximated by means of [4], and a way to find $G(p)$ for a given value of p could simply comprise a determination of $w(\sqrt{p})$. But due to the structure of the approximation of $w(z)$ the connection between $w(\sqrt{p})$ and $G(p)$ can be taken into account in a more efficient way.

Given a value of $p = pr + \mathrm{i}pi$ then $\sqrt{p} = sqrt(p) = sqrt(pr + \mathrm{i}pi) = x + \mathrm{i}y = z$ is computed according to a method [8], which is used in [3]. Then depending on the value of z the approximation of $G(p)$ is performed by one of two different methods:

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1) Small values of $|z|$:

The function $w(z)$ is written as

$$w(z) = e^{-z^2} + \frac{2i}{\sqrt{\pi}} z f\left(\frac{z^2}{5}\right)$$

where $f(t)$ is approximated using Lanczos' τ-method ([9], p. 489, ex. 5), but instead of using Chebyshev polynomials in the error term, it turns out to be better to use Legendre polynomials. In [2] section 3 the formulas are derived, and $f(t)$ is approximated as the ratio between two polynomials with real coefficients (of degree 10) in the complex variable $t = z^2/5 : f(t) \approx T(t)/N(t)$.

This means that the function $G(p)$ can be written

$$G(p) = 1 + i\sqrt{(\pi p)}\, e^{-p} - 10 \frac{\frac{p}{5} T\left(\frac{p}{5}\right)}{N\left(\frac{p}{5}\right)},$$

where $p/5\ T(p/5)$ and $N(p/5)$ are polynomials (with complex variable) which can be evaluated as in [4] using a procedure *PK* which is a simplified version of [1]; the method is given in [9], p. 16. In the ALGOL-text this part begins with the comment: Legendre approximation.

2) Large values of $|z|$:

The value of $w(z)$ is found as shown in [4] section 2.2 using a Gauss-Hermite quadrature, from which the function $G(p)$ is computed. In the ALGOL-text this part begins with the comment: Hermite quadrature.

Depending on the value of p, the following approximate execution times are obtained in the GIER ALGOL 4 system (where – for comparison – a call of the procedure *exp* (x) takes 4·4 msec ([10], p. 76)):

$0 \leqq \arg(p) \leqq \pi/2$: small $|p|$: approx. 100 msec
$0 \leqq \arg(p) \leqq \pi/2$: large $|p|$: approx. 50 msec
$\arg(p)$ not in the interval : approx. 10 msec.

No many-decimal table of the function $G(p)$ seems to exist, and consequently no direct test of the approximation has been possible. However, the accuracy can be estimated using the information about the computation of the function $w(z)$ ([4], section 4): $\mathrm{Re}(w(z))$ and/or $\mathrm{Im}(w(z))$ can have an absolute error up to $1{\cdot}5 \times 10^{-6}$, when z is in the neighbourhood of $1{\cdot}5 + i\,1{\cdot}5$, i.e. p near 5i. When $G(p)$ is determined from $w(z)$ (as shown above) the absolute error in $G(p)$ should not be greater than 10×10^{-6} when p is near 5i. For smaller values of $|p|$ the absolute error is smaller. For larger values of $|p|$ (or $|z|$) the relative error in $w(z)$ has not been determined, and the absolute error in $G(p)$ has been estimated as shown below. When $|p|$ is very small or very large the function $G(p)$ can easily be computed with high accuracy by means of simple formulas [11]. For $p = 0{\cdot}01, 0{\cdot}1, 50, 0{\cdot}01i, 0{\cdot}1i, 50i$ there was an error up to 2×10^{-8} in the results obtained by the procedure. This is in accordance with the fact that [4] is very accurate when $|z|$ is very small or very large. The procedure has also been tested in other ways (for example by comparing 441 pairs of values with the table [7]; for details, see [5] section 4.2.3), but the results of these tests can not change the following estimate of the accuracy of the approximation:

The absolute error in $G(p)$ is about $1 \times 10^{-5} - 1 \times 10^{-8}$.

```
boolean procedure Sommerfeld cox (pr, pi, gr, gi);
  value pr, pi;
  real pr, pi, gr, gi;
    comment This procedure computes the value of the Sommerfeld attenuation
    function: G(p).
    The parameters are:
    pr: real part of input p,
    pi: imaginary part of input p,
    gr: real part of output G(p),
    gi: imaginary part of output G(p),
    Sommerfeld cox: is true when 0 ≦ arg (p) ≦ phi/2, otherwise it is false;
if pr < 0 ∨ pi < 0
  then Sommerfeld cox := false
  else
    begin
      real x, y, M;
      Sommerfeld cox := true;
      M := pr↑2 + pi↑2;
      x := sqrt((sqrt(M) + pr)/2);
      y := if x = 0 then 0 else pi/2/x;
      if y > 1·7−0·2 × x ∨ y > 3·9 − x
        then
          begin comment Hermite quadrature;
              real p1, p2, p3, p4, p5, p6, n1, n2, n3, n4, n5, n6, a, b, T;
              M := y↑2;
              a := b := 0;
              for T := −x, x do
                begin
                  p1 := 0·31424 03763 + T;
                  p2 := 0·94778 83912 + T;
                  p3 := 1·59768 26352 + T;
```

```
            p4 := 2·27950 70805 + T;
            p5 := 3·02063 70251 + T;
            p6 := 3·88972 48979 + T;
            n1 := 0·18147 96822/(p1↑2 + M);
            n2 := 0·08291 72776 3/(p2↑2 + M);
            n3 := 0·01642 73320 3/(p3↑2 + M);
            n4 := 0·00124 31244 32/(p4↑2 + M);
            n5 := 0·00002 72908 9347/(p5↑2 + M);
            n6 := 0·00000 00846 24328 41/(p6↑2 + M);
            a  := a + n1 + n2 + n3 + n4 + n5 + n6;
            b  := -b + p1 × n1 + p2 × n2 + p3 × n3
            + p4 × n4 + p5 × n5 + p6 × n6
          end T;
        gr := 1 - 1·77245 38509 × (x × b + M × a);
        gi :=     1·77245 38 509 + (x × a - b) × y
    end Hermite quadrature
else
    begin comment Legendre approximation;
      real p1, p2, p3, n1, n2, t1, t2, T;
      procedure PK(pa, pb, a0, a1, a2, a3, a4, a5, a6, a7, a8, a9, a10);
        value a0, a1, a2, a3, a4, a5, a6, a7, a8, a9, a10;
        real pa, pb, a0, a1, a2, a3, a4, a5, a6, a7, a8, a9, a10;
      begin
        p3 := a9 + T × a10;
        p2 := a8 + T × p3 + M × a10;
        p1 := a7 + T × p2 + M × p3;
        p3 := a6 + T × p1 + M × p2;
        p2 := a5 + T × p3 + M × p1;
        p1 := a4 + T × p2 + M × p3;
        p3 := a3 + T × p1 + M × p2;
        p2 := a2 + T × p3 + M × p1;
        p1 := (a1 + T × p2 + M × p3)/5;
        pa := a0 + pr × p1 + M × p2;
        pb := pi × p1
      end PK;
      T  := 0·4 × pr;
      M := -0·04 × M;
      PK(t1, t2,
          0        ,   12096·51250,   -8488·78070,
      14448·00988,   -4495·93759,     3287·20821,
      -519·3045 ,       210·21    ,      -14·3     ,
          3·3      ,       0                        );
```

```
            PK(n1, n2,
            12096·51250, 31832·92763, 39914·35198,
            31537·26576, 17481·0636 ,  7151·3442 ,
             2207·205   ,   514·8     ,    89·1     ,
               11        ,     1                    );
            p3 := 10/(n1↑2 + n2↑2);
            p2 := cos (pi);
            p1 := sin (pi);
            T  := 1·77245 38509 × exp (−pr);
            gr := 1 + T × (x × p1 − y × p2) − p3 × (n1 × t1 + n2 × t2);
            gi :=     T × (x × p2 + y × p1) − p3 × (n1 × t2 − n2 × t1)
        end Legendre approximation
    end 0 ≦ arg (p) ≦ phi/2
finis Sommerfeld cox;
```

Test values.

pr	pi	gr	gi
0·01	0	0·980 132 803	0·175 481 762
0·1	0	0·812 814 910	0·507 160 572
50	0	−0·010 316 145	0·000 000 000
0	0·01	0·875 794 815	0·106 578 972
0	0·1	0·631 896 434	0·234 452 957
0	50	0·000 298 977	0·009 985 086
1	0	−0·076 159 008	0·652 049 327
10	0	−0·060 75	0·000 25
0	1	0·190 47	0·232 20
0	10	0·006 96	0·048 35
10	10	−0·024 34	0·029 16

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