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Computation of Sommerfeld's attenuation function

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This procedure computes the complex-valued Sommerfeld attenuation function, $G(p)$, which appears within the theory of propagation of electromagnetic waves [11]:

$$G(p) = 1 + i \sqrt{\pi p} e^{-p} \text{erfc} (-i \sqrt{p})$$

where

$$\text{erfc} (-i \sqrt{p}) = \frac{2}{\sqrt{\pi}} \int_{-i \sqrt{p}}^{\infty} e^{-t^2} \, dt,$$

provided that $0 \leq \arg(p) \leq \pi/2$. This function has been tabulated [7].

By means of the function $w(z)$ [6], defined by

$$w(z) = e^{-z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{-r^2} \, dr\right) = e^{-z^2} \text{erfc} (-iz)$$

the function $G(p)$ can be expressed as

$$G(p) = 1 + i \sqrt{\pi p} w(\sqrt{p}).$$

The function $w(z)$ can be approximated by means of [4], and a way to find $G(p)$ for a given value of $p$ could simply comprise a determination of $w(\sqrt{p})$. But due to the structure of the approximation of $w(z)$ the connection between $w(\sqrt{p})$ and $G(p)$ can be taken into account in a more efficient way.

Given a value of $p = pr +opi$ then $\sqrt{p} = \sqrt{\text{sqrt}(pr +opi)} = x + iy = z$ is computed according to a method [8], which is used in [3]. Then depending on the value of $z$ the approximation of $G(p)$ is performed by one of two different methods:

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1) Small values of $|z|$

The function $w(z)$ is written as

$$w(z) = e^{-z^2} + \frac{2i}{\sqrt{\pi}} z f\left(\frac{z^2}{5}\right)$$

where $f(t)$ is approximated using Lanczos' \( \tau \)-method ([9], p. 489, ex. 5), but instead of using Chebyshev polynomials in the error term, it turns out to be better to use Legendre polynomials. In [2] section 3 the formulas are derived, and $f(t)$ is approximated as the ratio between two polynomials with real coefficients (of degree 10) in the complex variable $t = z^2/5 : f(t) \approx T(t)/N(t)$.

This means that the function $G(p)$ can be written

$$G(p) = 1 + i \sqrt(\pi p) e^{-p} - 10 \frac{p}{5} T\left(\frac{p}{5}\right)$$

where $p/5 T(p/5)$ and $N(p/5)$ are polynomials (with complex variable) which can be evaluated as in [4] using a procedure $PK$ which is a simplified version of [1]; the method is given in [9], p. 16. In the ALGOL-text this part begins with the comment: Legendre approximation.

2) Large values of $|z|$

The value of $w(z)$ is found as shown in [4] section 2.2 using a Gauss-Hermite quadrature, from which the function $G(p)$ is computed. In the ALGOL-text this part begins with the comment: Hermite quadrature.

Depending on the value of $p$, the following approximate execution times are obtained in the GIER ALGOL 4 system (where — for comparison — a call of the procedure $\text{exp}(x)$ takes 4.4 msec ([10], p. 76)):

- $0 \leq \arg(p) \leq \pi/2$: small $|p|$: approx. 100 msec
- $0 \leq \arg(p) \leq \pi/2$: large $|p|$: approx. 50 msec
- $\arg(p)$ not in the interval: approx. 10 msec.

No many-decimal table of the function $G(p)$ seems to exist, and consequently no direct test of the approximation has been possible. However, the accuracy can be estimated using the information about the computation of the function $w(z)$ ([4], section 4): Re($w(z)$) and/or Im($w(z)$) can have an absolute error up to $1 \cdot 10^{-6}$, when $z$ is in the neighbourhood of $1 \cdot 5 + i \cdot 1 \cdot 5$, i.e. $p$ near 5i. When $G(p)$ is determined from $w(z)$ (as shown above) the absolute error in $G(p)$ should not be greater than $10 \times 10^{-6}$ when $p$ is near 5i. For smaller values of $|p|$ the absolute error is smaller. For larger values of $|p|$ (or $|z|$) the relative error in $w(z)$ has not been determined,
and the absolute error in $G(p)$ has been estimated as shown below. When $|p|$ is very small or very large the function $G(p)$ can easily be computed with high accuracy by means of simple formulas [11]. For $p = 0.01, 0.1, 50, 0.01i, 0.1i, 50i$ there was an error up to $2 \times 10^{-8}$ in the results obtained by the procedure. This is in accordance with the fact that [4] is very accurate when $|z|$ is very small or very large.

The procedure has also been tested in other ways (for example by comparing 441 pairs of values with the table [7]; for details, see [5] section 4.2.3), but the results of these tests can not change the following estimate of the accuracy of the approximation:

The absolute error in $G(p)$ is about $1 \times 10^{-5} - 1 \times 10^{-8}$.

```plaintext
boolean procedure Sommerfeld cox (pr, pi, gr, gi);
value pr, pi;
real pr, pi, gr, gi;
comment This procedure computes the value of the Sommerfeld attenuation function: $G(p)$.
The parameters are:
pr: real part of input $p$,
pi: imaginary part of input $p$,
gr: real part of output $G(p)$,
gi: imaginary part of output $G(p)$,
Sommerfeld cox: is true when $0 \leq \arg(p) \leq \phi/2$, otherwise it is false;
if pr < 0 \lor pi < 0
then Sommerfeld cox := false
else
begin
real x, y, M;
Sommerfeld cox := true;
M := pr\^2 + pi\^2;
x := sqrt((sqrt(M) + pr)/2);
y := if x = 0 then 0 else pi/2/x;
if y > 1.7 - 0.2 \times x \lor y > 3.9 - x
then
begin comment Hermite quadrature;
real p1, p2, p3, p4, p5, p6, n1, n2, n3, n4, n5, n6, a, b, T;
M := y\^2;
a := b := 0;
for T := -x, x do
begin
p1 := 0.31424 03763 + T;
p2 := 0.94778 83912 + T;
p3 := 1.59768 26352 + T;
end
```

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\[ p_4 := 2.27950 \times 10^5 + T; \]
\[ p_5 := 3.02063 \times 10^5 + T; \]
\[ p_6 := 3.88972 \times 10^5 + T; \]
\[ n_1 := 0.18147 \times 10^2 / (p_1^2 + M); \]
\[ n_2 := 0.08291 \times 10^3 / (p_2^2 + M); \]
\[ n_3 := 0.01642 \times 10^3 / (p_3^2 + M); \]
\[ n_4 := 0.00124 \times 10^3 / (p_4^2 + M); \]
\[ n_5 := 0.00002 \times 10^3 / (p_5^2 + M); \]
\[ n_6 := 0.00000 \times 10^3 / (p_6^2 + M); \]
\[ a := a + n_1 + n_2 + n_3 + n_4 + n_5 + n_6; \]
\[ b := -b + p_1 \times n_1 + p_2 \times n_2 + p_3 \times n_3 + p_4 \times n_4 + p_5 \times n_5 + p_6 \times n_6; \]
end T;

\[ g_r := 1 - 1.77245 \times 10^5 \times (x \times b + M \times a); \]
\[ g_i := 1.77245 \times 10^5 \times (x \times a - b) \times y \]
end Hermite quadrature

else
begin comment Legendre approximation;
real p_1, p_2, p_3, n_1, n_2, t_1, t_2, T;
procedure PK(pa, pb, a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_10);
value a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_10;
real pa, pb, a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_10;
begin
\[ p_3 := a_9 + T \times a_{10}; \]
\[ p_2 := a_8 + T \times p_3 + M \times a_{10}; \]
\[ p_1 := a_7 + T \times p_2 + M \times p_3; \]
\[ p_3 := a_6 + T \times p_1 + M \times p_2; \]
\[ p_2 := a_5 + T \times p_3 + M \times p_1; \]
\[ p_1 := a_4 + T \times p_2 + M \times p_3; \]
\[ p_3 := a_3 + T \times p_1 + M \times p_2; \]
\[ p_2 := a_2 + T \times p_3 + M \times p_1; \]
\[ p_1 := (a_1 + T \times p_2 + M \times p_3)/5; \]
\[ pa := a_0 + p_1 \times p_1 + M \times p_2; \]
\[ pb := p_1 \times p_1 \]
end PK;
T := 0.4 \times pr;
M := -0.04 \times M;
PK(t_1, t_2,
\begin{array}{cccc}
0 & 12096.51250 & -8488.78070 & 14448.00988 & -4495.93759 & 3287.20821 & -519.30458 & 210.218 & -14.3 & 3.3 & 0 \\
\end{array});
\[ PK(n_1, n_2, \\
12096 \cdot 51250, 31832 \cdot 92763, 39914 \cdot 35198, \\
31537 \cdot 26576, 17481 \cdot 0636, 7151 \cdot 3442, \\
2207 \cdot 205, 514 \cdot 8, 89 \cdot 1, \\
11, 1 \); \]

\[ p_3 := 10 / (n_1 t_2 + n_2 t_2); \]

\[ p_2 := \cos (p_3); \]

\[ p_1 := \sin (p_3); \]

\[ T := 1.77245 \cdot 38509 \times \exp (-p_1); \]

\[ g_r := 1 + T \times (x \times p_1 - y \times p_2) - p_3 \times (n_1 \times t_1 + n_2 \times t_2); \]

\[ g_i := T \times (x \times p_2 + y \times p_1) - p_3 \times (n_1 \times t_2 - n_2 \times t_1); \]

\end Legendre approximation

\end 0 \leq \arg (p) \leq \phi / 2

\textit{finis Sommerfeld cox};

**Test values.**

<table>
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<tr>
<th>pr</th>
<th>pi</th>
<th>g_r</th>
<th>g_i</th>
</tr>
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<tbody>
<tr>
<td>0.01</td>
<td>0</td>
<td>0.980 132 803</td>
<td>0.175 481 762</td>
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<td>0.1</td>
<td>0</td>
<td>0.812 814 910</td>
<td>0.507 160 572</td>
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<td>0</td>
<td>-0.010 316 145</td>
<td>0.000 000 000</td>
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<tr>
<td>0</td>
<td>0.01</td>
<td>0.875 794 815</td>
<td>0.106 578 972</td>
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<tr>
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<td>0.1</td>
<td>0.631 896 434</td>
<td>0.234 452 957</td>
</tr>
<tr>
<td>0</td>
<td>50</td>
<td>0.000 298 977</td>
<td>0.009 985 086</td>
</tr>
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<td>1</td>
<td>0</td>
<td>-0.076 159 008</td>
<td>0.652 049 327</td>
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<td>10</td>
<td>0</td>
<td>-0.060 75 75</td>
<td>0.000 25</td>
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<tr>
<td>0</td>
<td>1</td>
<td>0.190 47</td>
<td>0.232 20</td>
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<td>0.006 96</td>
<td>0.048 35</td>
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<tr>
<td>10</td>
<td>10</td>
<td>-0.024 34</td>
<td>0.029 16</td>
</tr>
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References

[6] Faddeyeva, V. N. and N. M. Terent’ev: Tables of Values of the Function \( w(z) = e^{-z^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{it^2} dt \right) \) for Complex Argument. Oxford: Pergamon Press. 1961.