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SOME $L_2$ – ERROR ESTIMATES FOR SEMI-VARIATIONAL METHOD APPLIED TO PARABOLIC EQUATIONS

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The semi-variational method for parabolic equations [1] presents a sequence of approximations to the solution with an increasing accuracy measured in the time-increment. The first semi-variational approximation coincides with the Crank-Nicolson Galerkin procedure [2], [3], which is second order correct in time.

In a recent article [3], Dupont proved some estimates of the $L_2$-norms of the errors for the Crank-Nicolson Galerkin method applied to linear equations involving a non-selfadjoint time-independent operator of the second order. The purpose of this paper is to present similar estimates for the second semi-variational approximation. The approach of [3] has to be slightly generalized to prove that the second approximation is fourth order correct in time even in case of non-selfadjoint operators.

The $L_2$-estimates differ from those of [1], [2] not only by higher accuracy in space and by an explicit dependence on the given data but also by different regularity hypotheses on the solution of the parabolic problem.

1. NOTATION, PARABOLIC REGULARITY

Let $\Omega$ be a bounded domain in the n-dimensional Euclidean space $R^n$ with a smooth boundary $\Gamma \in C^\infty$.

$H^s(\Omega)$, with $s$ non-negative integer, will denote the Sobolev space of all functions in $L_2(\Omega)$, whose distribution derivatives up to the order $s$ are also in $L_2(\Omega)$. The norm in $H^s(\Omega)$ will be defined through

$$\|u\|_s^2 = \sum_{|\alpha| \leq s} \|D^\alpha u\|^2,$$

where $\alpha$ is the multi-index,

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \ldots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n,$$

and the index $s = 0$ is omitted.
The scalar product in \( L_2(\Omega) \) is denoted by

\[
(u, v) = \int_{\Omega} uv \, dx
\]

and the norm

\[
\|u\| = (u, u)^{1/2}.
\]

\( H^{-1} \) denotes the space of linear continuous functionals on \( H^1(\Omega) \), i.e., \( H^{-1} = (H^1)' \), with the norm

\[
\|f\|_{-1} = \sup_{\|v\|_{1} \leq 1} \frac{|\langle f, v \rangle|}{\|v\|_{1}},
\]

where \( \langle f, v \rangle \) is the extension of \((f, v)\):

\[
\langle f, v \rangle = (f, v) \quad \text{if} \quad f \in L_2(\Omega).
\]

We use also the following notations

\[
\|u\|_{L^2(X)}^2 = \int_0^T \|u(t)\|_X^2 \, dt,
\]

for any function \( u(t) \), mapping the interval \( <0, T> \) into a normed space \( X \).

We shall consider the parabolic equation

\[
\frac{\partial u}{\partial t} + Au = f \quad \text{on} \quad \Omega \times (0, T), \quad T < \infty,
\]

with the initial condition

\[
u(\cdot, 0) = \varphi \quad \text{on} \quad \Omega
\]

and the Neumann's boundary condition

\[
a_{ij} \frac{\partial u}{\partial x_i} v_j = 0 \quad \text{on} \quad \Gamma \times (0, T),
\]

where \( A \) is the uniformly elliptic operator

\[
Au = -\frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + b_i \frac{\partial u}{\partial x_i} + a_0 u
\]

and \( v_j \) are the components of the unit outward normal to \( \Gamma \). The repeated index implies summation over the range 1, 2, ..., \( n \). The coefficients \( a_{ij}(x) \) form a \( n \times n \) symmetric positive definite matrix for each \( x = (x_1, x_2, \ldots, x_n) \in \Omega \). All coefficients
\( a_{ij}, b_j, a_0 \) belong to \( C^\infty(\Omega) \), (i.e., they can be extended to be infinitely differentiable on \( \mathbb{R}^n \), being independent of \( t \). The right-hand side of (1.4) \( f \) is a mapping of \( \left< 0, T \right> \) into \( H^{-1} \), which is continuous at \( t = 0 \). The function \( \varphi \) belongs to \( L_2(\Omega) \).

We introduce the following bilinear form on \( H^1 \)

\[
[ u, v ]_A = \left( a_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) + \left( b_j \frac{\partial u}{\partial x_j} + a_0 u, v \right).
\]

Note that the form (1.7) is continuous but not symmetric unless the coefficients \( b_j \) vanish identically.

Let \( \mathcal{M} \) be a finite-dimensional subspace of \( H^1(\Omega) \), spanned by elements \( v_1, v_2, \ldots, v_N \).

The first semi-variational approximation \( u^{(1)}(x, t) \) (cf. [1]) is called Crank-Nicolson-Galerkin approximation [2], being determined by the equations

\[
(1.8) \quad \frac{1}{\tau} (U_{m+1} - U_m, V) + \frac{1}{2} [ U_{m+1} + U_m, V ]_A = \frac{1}{2} \left< f_{m+1} + f_m, V \right>, \quad V \in \mathcal{M}, \quad m = 0, 1, 2, \ldots, M - 1; \quad U_s \in \mathcal{M}, \quad s \geq 0;
\]

where

\[
M = T/\tau, \quad U_0 = u^{(1)}(\cdot, s\tau), \quad f_s = f(s\tau)
\]

and

\[
(1.9) \quad (U_0 - \varphi, V) = 0, \quad V \in \mathcal{M}.
\]

In the subintervals \( \langle m\tau, m\tau + \tau \rangle \), \( u^{(1)}(x, t) \) is defined as the linear interpolate of \( U_m, U_{m+1} \).

The second semi-variational approximation \( u^{(2)}(x, t) \) [1] is determined by the system of equations (1.9) and

\[
(1.10) \quad \frac{1}{\tau} (U_{m+1} - U_m, V) + \frac{1}{2} [ U_m + 4U_{m+1/2} + U_{m+1}, V ]_A =
= \frac{1}{2} \left< f_m + 4f_{m+1/2} + f_{m+1}, V \right>,
\]

\[
\frac{4}{\tau} (U_m - 2U_{m+1/2} + U_{m+1}, V) + [ U_{m+1} - U_m, V ]_A = \left< f_{m+1} - f_m, V \right>,
\]

\( V \in \mathcal{M}, \quad m = 0, 1, 2, \ldots, M - 1; \quad U_s \in \mathcal{M}, \quad s \geq 0; \quad U_s = u^{(2)}(\cdot, s\tau).
\]

In the subintervals \( \langle m\tau, m\tau + \tau \rangle \) \( u^{(2)}(x, t) \) is defined as the quadratic interpolate of \( U_m, U_{m+1/2}, U_{m+1} \). The system (1.8) for \( U_{m+1} \) and (1.10) for \( U_{m+1/2}, U_{m+1} \), respectively, possesses a unique solution for sufficiently small \( \tau \) and any \( m = 0, 1, \ldots, M - 1 \).
In order to prove this assertion let us note first, that there exist positive constants $\lambda, \alpha$ such that

\[(1.11) \quad [v, v]_A + \lambda \|v\|^2 \geq \alpha \|v\|^2_1, \quad v \in H^1(\Omega).\]

In fact,

\[
\left( a_{ij} \frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) \geq c_0 |v|_1^2,
\]

\[
\left( b_j \frac{\partial v}{\partial x_j} + a_{0i} v, v \right) \leq C_1 |v|_1 \|v\| + C_2 \|v\|^2 \leq C_1 \varepsilon |v|_1^2 + \left( C_1 \frac{1}{4\varepsilon} + C_2 \right) \|v\|^2,
\]

where

\[
|v|_1^2 = \sum_{i=1}^n \left\| \frac{\partial v}{\partial x_i} \right\|^2.
\]

Consequently, we may write

\[
[v, v]_A + \lambda \|v\|^2 \geq (c_0 - C_1 \varepsilon) |v|_1^2 + \left( \lambda - C_2 - C_1 \frac{1}{4\varepsilon} \right) \|v\|^2
\]

and choosing $\varepsilon, \lambda$ such that

\[
c_0 - C_1 \varepsilon \geq c_0/2, \quad \lambda - C_2 - C_1 \frac{1}{4\varepsilon} \geq c_0/2,
\]

we obtain (1.11) with $c_0/2 \equiv \alpha$.

The system (1.8) for $U_{m+1}$ can be rewritten as follows

\[(1.8') \quad [U_{m+1}, V]_A + \frac{2}{\tau} (U_{m+1}, V) = Y_m,
\]

where $Y_m$ is a known vector. The solution of the corresponding homogeneous system satisfies

\[(1.12) \quad [U_{m+1}, U_{m+1}]_A + \frac{2}{\tau} \|U_{m+1}\|^2 = 0.
\]

From (1.11) and (1.12) it follows that the matrix of (1.8') is regular, if $2/\tau \geq \lambda$.

The system (1.10) for $U_{m+1/2}, U_{m+1}$ can be rewritten in the following equivalent form

\[(1.10')_1 \quad \frac{1}{\tau} (U_{m+1}, V) + [W, V]_A = T_m,
\]

\[(1.10')_2 \quad \frac{4}{\tau} (-3W + \frac{3}{2}U_{m+1}, V) + [U_{m+1}, V]_A = Z_m,
\]

where $T_m$ and $Z_m$ are known vectors and

\[
W = \frac{1}{6} U_{m+1} + \frac{3}{3} U_{m+1/2}.
\]
Let us consider the solution \( W, U_{m+1} \) of the corresponding homogeneous system and insert \( V = 2W \) into (1.10'), and \( V = U_{m+1} \) into (1.10')to obtain

\[
\tau^{-1}(U_{m+1}, 12W) + 12[W, W]_A = 0, \\
4\tau^{-1}(U_{m+1}, -3W + \frac{3}{2}U_{m+1}) + [U_{m+1}, U_{m+1}]_A = 0.
\]

The sum of (1.13) and (1.14) yields

\[
6\tau^{-1}\|U_{m+1}\|^2 + [U_{m+1}, U_{m+1}]_A + 12[W, W]_A = 0.
\]

Inserting \( V = U_{m+1} \) into (1.10'), we obtain

\[
\|U_{m+1}\|^2 = -\tau[W, U_{m+1}]_A \leq C_1\tau\|W\|_1\|U_{m+1}\|_1.
\]

Inserting \( V = W \) into (1.10'), we obtain

\[
12\tau^{-1}\|W\|^2 = 6\tau^{-1}(U_{m+1}, W) + [U_{m+1}, W]_A.
\]

Hence it follows, by virtue of (1.16), that

\[
12\delta\|W\|^2 \leq 6\delta\|U_{m+1}\|\|W\| + \lambda C_1\|U_{m+1}\|_1\|W\|_1 \leq \lambda C_1\|U_{m+1}\|_1\|W\|_1 \leq \frac{1}{2}(\lambda C_1\tau + \frac{3}{4}C_1\frac{\lambda}{\delta}\|W\|^2).
\]

If we put \( \varepsilon = \delta/\lambda \), then

\[
12\delta\|W\|^2 \leq 6\delta\|W\|^2 + C_1\tau\left(\frac{1}{2} + \frac{3}{4}\right)(\|U_{m+1}\|^2 + \|W\|^2).
\]

For \( 6/\tau \geq \lambda \), from (1.11) and (1.17) it follows that

\[
[U_{m+1}, U_{m+1}]_A + 6\tau^{-1}\|U_{m+1}\|^2 + 12([W, W]_A + \lambda\|W\|^2) - 12\lambda\|W\|^2 \geq \left(x - \tau C_1\left(\frac{1}{2} + \frac{3}{4}\right)\right)\|U_{m+1}\|^2 + \left(x - \tau C_1\left(\frac{1}{2} + \frac{3}{4}\right)\right)\|W\|^2.
\]

Consequently, for

\[
\tau < \min\left\{2C_1^{-1}\left(\frac{1}{2} + \frac{3}{4}\right)^{-1}, 6/\lambda\right\}
\]

(1.15 and (1.18) imply \( U_{m+1} = W = \Theta \). Hence the matrix of (1.10) is regular, Q.E.D.

We shall assume that the space \( M \) belongs to a family \( \{M_h\} \) \((0 < h \leq 1)\) of subspaces of \( H^1(\Omega) \), which satisfy the following approximation assumptions:

There is a constant \( C_0 \) and an integer \( r \geq 1 \), both independent of \( h \), such that for \( 1 \leq s \leq 2r \) and \( v \in H^r(\Omega) \)

\[
\inf_{\chi \in M_h} \{\|v - \chi\| + h\|v - \chi\|_1\} \leq C_0h^s\|v\|_s.
\]
A parabolic regularity result will be formulated in terms of the following norms

\[
\|u\|_{W^s} = \sum_{j=0}^{s} \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^\infty(H^{2s-j})} + \sum_{j=0}^{s+1} \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^2(H^{2s+1-j})}, \quad s \geq 0,
\]

\[
\|u\|_{W^{s-1}} = \left\| u \right\|_{L^2(H^{-1})},
\]

(1.12)

\[
\|u\|_{G^s} = \left\| u^{(0,0)} \right\|_{s} + \left\| \frac{\partial u}{\partial t} + Au \right\|_{W^{s-1}}, \quad s \geq 0.
\]

\[G^s\] will denote the completion of the set

\[
\left\{ u \in C^\infty(\bar{\Omega} + \langle 0, T \rangle) \mid a_{ij} \frac{\partial u}{\partial x_j} = 0 \right\}
\]

with respect to the norm \[\| \cdot \|_{G^s}\].

Moreover, we introduce the set

\[D^s = \{ [\varphi, f] \mid \exists u \in G^s \text{ such that } u^{(0,0)} = \varphi \text{ on } \Omega \text{ and } \frac{\partial u}{\partial t} + Au = f \text{ on } \Omega \times (0, T) \}.
\]

Thus \[D^s\] is the set of data which give solutions in \[G^s\].

**Lemma 1. (Parabolic regularity).**

*For \( s \geq 0 \), there is a constant \( C(s) \) such that*

\[
\|u\|_{W^s} \leq C(s)\|u\|_{G^s}, \quad u \in G^s.
\]

The proof is given in [3]. It uses the usual energy estimate, Gronwall's lemma and elliptic regularity.

## 2. ERROR ESTIMATES

For completeness, we present here the error estimate also for the first approximations, which was proved by Dupont [3].

**Theorem 1. (Dupont).** Let \( U_m \) be the values of the Crank-Nicolson Galerkin approximation (1.8), (1.9), \( u_m \) the solution of (1.4), (1.5) at \( t = mt \). Let \( s = \max(2, r) \) and the pair \([\varphi, f] \in D^s\).

Then such positive constants \( \tau_0, C_1, C_2 \) exist that for \( 0 < \tau \leq \tau_0 \)

\[
\max_{0 \leq m \leq M} \| U_m - u_m \| \leq C_1 \{ h^{2r} \| u \|_{W^s} + \tau^2 \| u \|_{W^s} \} \leq C_2 (h^{2r} + \tau^2) \{ \| \varphi \|_{2s} + \| f \|_{W^{s-1}} \}.
\]

The main result of the present paper is the following.
Theorem 2. Let $U_m$ be the values of the second semi-variational approximation (1.9), (1.10), $u_m$ the solution of (1.4), (1.5) at $t = mt$. Let $s = \max (5, r)$ and the pair $[\varphi, f] \in D'$. Then such positive constants $\tau_1, C_3, C_4$ exist that for $0 < \tau \leq \tau_1$

\[
\max_{0 \leq m \leq M} \|U_m - u_m\| \leq C_3 \{h^{2r}\|u\|_{W^r} + \tau^4\|u\|_{W^1}\} \leq C_4(h^{2r} + \tau^4) \{\|\varphi\|_{2s} + \|f\|_{W^{s-1}}\}.
\]

Proof. First we shall define a projection of the solution $u$ of (1.4), (1.5) into the subspace $H_h$. For each $t \in (0, T)$ let $W(t) \in H_h$ be determined by

\[
[u - W, V] + \lambda(u - W, V) = 0, \quad V \in H_h,
\]

where $\lambda$ is a sufficiently large constant such that (1.11) holds.

We shall need the following

Lemma 2. There is a constant $C$, independent of $h$ and $u$, such that for each $t \in (0, T)$ and for $2 \leq s \leq 2r$, $r \geq 2$, $u \in H^s(\Omega)$

\[
\|W - u\| + h^{-1}\|W - u\|_{-1} \leq Ch^s\|u\|_s.
\]

The proof of Lemma 2 can be found e.g. in [4]. The immediate consequence of this Lemma is

Lemma 3. There is a constant $C$ such that if $u \in G', r \geq 2$ and $\eta = W - u$, then

\[
\|\eta\|_{L^\infty(L^2)} + \left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(H^{-1})} \leq C h^{2r}\|u\|_{W^r}.
\]

Proof. From Lemma 1 we conclude that $u \in W^r$. Applying Lemma 2 to $\eta$ and $\frac{\partial \eta}{\partial t}$, we obtain

\[
\|\eta\| \leq C h^{2r}\|u\|_{2r}, \quad \left\|\frac{\partial \eta}{\partial t}\right\|_1 \leq C h^{2r}\left\|\frac{\partial u}{\partial t}\right\|_{2r-1}
\]

and from there (2.5) follows.

Denote

\[
\delta \sigma_m = \sigma_{m+1} - \sigma_m, \quad \sigma_m^\wedge = \frac{1}{10}(\sigma_m + 4\sigma_{m+1/2} + \sigma_{m+1}),
\]

\[
\Delta \sigma_m = \sigma_m - 2\sigma_{m+1/2} + \sigma_{m+1} = -3\sigma_{m}^\wedge + \frac{3}{5} (\sigma_m + \sigma_{m+1} + \frac{1}{2} (\sigma_m + \sigma_{m+1}),
\]

\[
\partial_m = U_m - W_m, \quad \eta_m = W_m - u_m, \quad z_m = U_m - u_m.
\]
We can see that the solution \( u \) of (1.4), (1.5) satisfies the equation

\[
(2.8) \quad \left( \frac{\partial u}{\partial t} , v \right) + [u, v]_A = \langle f, v \rangle , \quad 0 \leq t \leq T , \quad v \in H^1(\Omega).
\]

In fact, \( u \in G^s \subset W^s \) with \( s \geq 5 \), therefore

\[
\left\| u \right\|_{L^2(H^2)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(H^2)} < \infty.
\]

Then \( u \) and \( \frac{\partial u}{\partial t} \) equal almost everywhere to a continuous mapping of the interval \( (0, T) \) into \( H^2 \) and \( L^2 \), respectively (see e.g. [5], p. 5) Accepting this continuity, we come to (2.8) even for \( t = 0 \) by the limit procedure.

Thus we may write

\[
(2.9) \quad \frac{1}{\tau} (\delta u_m, v) + [u^m, v]_A = \langle f_m, v \rangle + (\varrho_m, v) , \quad v \in H^1,
\]

where

\[
(2.10) \quad \varrho_m = \frac{1}{\tau} \delta u_m - \left( \frac{\partial u}{\partial t} \right)^m_m , \quad m = 0, 1, \ldots, M - 1,
\]

\[
(2.11) \quad \frac{4}{\tau} (Au_m, v) + [\delta u_m, v]_A = \langle \delta f_m, v \rangle + (\zeta_m, v) , \quad v \in H^1,
\]

where

\[
(2.12) \quad \zeta_m = \frac{4}{\tau} Au_m - \delta \left( \frac{\partial u}{\partial t} \right)^m_m , \quad m = 0, 1, \ldots, M - 1.
\]

Subtracting (2.9) and (2.11) from the corresponding equations (1.10), we derive

\[
(2.13) \quad \frac{1}{\tau} (\delta z_m, V) + [z^m, V]_A = -(\varrho_m, V) , \quad V \in \mathcal{H}_h,
\]

\[
(2.14) \quad \frac{4}{\tau} (Az_m, V) + [\delta z_m, V]_A = -(\zeta_m, V) , \quad V \in \mathcal{H}_h.
\]

Inserting \( z_m = \vartheta_m + \eta_m \), \( V = \vartheta^\omega \) and \( V = \delta \vartheta_m \), respectively, we obtain

\[
(2.15) \quad \frac{1}{\tau} (\delta \vartheta_m, \vartheta^\omega_m) + [\vartheta^\omega_m, \vartheta^\omega_m]_A = \frac{-1}{\tau} (\delta \eta_m, \vartheta^\omega_m) - [\eta^\omega_m, \vartheta^\omega_m]_A - (\varrho_m, \vartheta^\omega_m),
\]

\[
(2.16) \quad \frac{4}{\tau} (A \vartheta_m, \delta \vartheta_m) + [\delta \vartheta_m, \delta \vartheta_m]_A = -\frac{4}{\tau} (A \eta_m, \delta \vartheta_m) - [\delta \eta_m, \delta \vartheta_m]_A - (\zeta_m, \delta \vartheta_m).
\]
If we multiply (2.15) by 12 and add to (2.16), we may write, using also (2.3)

\[
(2.17) \quad \frac{6}{\tau} (\| \mathcal{G}_{m+1} \|^2 - \| \mathcal{G}_m \|^2) + \left[ \delta \mathcal{G}_m, \delta \mathcal{G}_m \right]_A + 12 \left[ \mathcal{G}_m, \mathcal{G}_m \right]_A =
\]
\[
= - \frac{12}{\tau} (\delta \eta_m, \mathcal{G}_m) + 12 \lambda (\delta \mathcal{G}_m, \mathcal{G}_m) - 12 (\mathcal{G}_m, \mathcal{G}_m) -
\]
\[
- \frac{4}{\tau} (\Delta \eta_m, \delta \mathcal{G}_m) + \lambda (\delta \eta_m, \delta \mathcal{G}_m) - (\zeta_m, \delta \mathcal{G}_m).
\]

With the use of both (2.4) and the extension of the scalar product in $L_2$, according to (1.2), we obtain

\[
(2.18) \quad \frac{6}{\tau} (\| \mathcal{G}_{m+1} \|^2 - \| \mathcal{G}_m \|^2) + \alpha \| \delta \mathcal{G}_m \|_1^2 + 12 \alpha \| \mathcal{G}_m \|^2 - 12 \lambda \| \mathcal{G}_m \|^2 \leqleq C_{1} \epsilon (\| \delta \mathcal{G}_m \|_1^2 + \| \mathcal{G}_m \|_1^2) + C_{2} \psi_m,
\]

where $\epsilon$ is an arbitrary small positive constant and

\[
(2.19) \quad \psi_m = \left\| \frac{1}{\tau} \Delta \eta_m \right\|_{-1}^2 + \left\| \frac{1}{\tau} \delta \eta_m \right\|_{-1}^2 + \left\| \eta_m \right\|_{-1}^2 + \left\| \mathcal{G}_m \right\|_{-1}^2 - (\zeta_m, \delta \mathcal{G}_m).
\]

Let us derive an estimate for $\| \mathcal{G}_m \|$. To this end, insert $\nu = \mathcal{G}_m$ and $\omega_m = \mathcal{G}_m + \eta_m$ in (2.14):

\[
(2.20) \quad \frac{4}{\tau} \left( -3 \mathcal{G}_m^\wedge + \frac{3}{2} (\mathcal{G}_m + \mathcal{G}_{m+1}, \mathcal{G}_m^\wedge) \right) + \left[ \delta \mathcal{G}_m, \mathcal{G}_m^\wedge \right]_A =
\]
\[
= - \frac{4}{\tau} (\Delta \eta_m, \mathcal{G}_m^\wedge) - \left[ \delta \eta_m, \mathcal{G}_m^\wedge \right]_A - (\zeta_m, \mathcal{G}_m^\wedge).
\]

Using moreover (2.3), we arrive at

\[
\left[ \delta \eta_m, \mathcal{G}_m^\wedge \right]_A = \lambda (\delta \eta_m, \mathcal{G}_m^\wedge) \leq \lambda \| \delta \eta_m \|_{-1} \| \mathcal{G}_m \|_{1}.
\]

Consequently, (2.20) yields

\[
12 \| \mathcal{G}_m \|^2 \leq 6 (\| \mathcal{G}_m \| + \| \mathcal{G}_{m+1} \|) \| \mathcal{G}_m \| + \tau C_{4} \| \delta \mathcal{G}_m \|_{1} \| \delta \mathcal{G}_m \|_{1} +
\]
\[
+ 4 \| \Delta \eta_m \|_{-1} \| \mathcal{G}_m \|_{1} + \lambda \tau e \| \delta \mathcal{G}_m \|_{-1} \| \mathcal{G}_m \|_{1} + \| \tau \mathcal{G}_m \| \| \mathcal{G}_m \| \leqleq
\]
\[
\leq 6 \| \mathcal{G}_m \|_{1}^2 + C (\| \mathcal{G}_m \|^2 + \| \mathcal{G}_{m+1} \|^2) + C_{4} \tau \frac{4}{2} (\| \delta \mathcal{G}_m \|_{1}^2 + \| \mathcal{G}_m \|_{1}^2) +
\]
\[
+ 4 \| \mathcal{G}_m \|_{1}^2 + 4 C \| \Delta \eta_m \|_{-1}^2 + \lambda \tau e \| \mathcal{G}_m \|_{1}^2 + C \lambda \tau \| \Delta \eta_m \|_{-1}^2 +
\]
\[
+ \epsilon \| \mathcal{G}_m \|^2 + C \| \tau \mathcal{G}_m \|_{1}^2.
\]
For sufficiently small $\varepsilon$ we have the estimate

$$(2.21) \|g_m^{\wedge}\|^2 \leq C \left\{ \|g_m\|^2 + \|g_{m+1}\|^2 + \|\Delta \eta_m\|^2_{-1} + \frac{1}{\tau} \|\delta \eta_m\|^2_{-1} + \|\tau \xi_m\|^2 \right\} +$$

$$+ C_1(\tau + \varepsilon + \tau \varepsilon) \|g_m^{\wedge}\|^2_{1} + \frac{1}{2} C_4 \|\delta \eta_m\|^2_{7}.$$  

Let us use (2.21) in (2.18) to obtain

$$(2.22) \frac{6}{\tau} (\|g_{m+1}\|^2 - \|g_m\|^2) + \|g_m\|^2 + 12\varepsilon \|g_m^{\wedge}\|^2_{1} \leq$$

$$\leq C_1 \varepsilon (\|\delta \eta_m\|^2_{1} + \|g_m^{\wedge}\|^2_{1}) + C_2 \psi_m +$$

$$+ C_3 \left\{ \|g_m\|^2 + \|g_{m+1}\|^2 + (\tau + \varepsilon + \tau \varepsilon) \|g_m^{\wedge}\|^2_{1} + \tau \|\delta \eta_m\|^2_{1} + \|\tau \xi_m\|^2 \right\}.$$  

If $\varepsilon$ and $\tau$ are sufficiently small, we have the inequality

$$(2.23) \frac{1}{\tau} (\|g_{m+1}\|^2 - \|g_m\|^2) + \|g_m\|^2 \leq C \left\{ \|g_m\|^2 + \|g_{m+1}\|^2 + \|\tau \xi_m\|^2 + \psi_m \right\},$$

where the term $\|g_m\|^2$ was added on both sides.

**Lemma 4.** (Discrete analogue of Gronwall's inequality).

Let

$$(2.24) \frac{1}{\tau} \left( \|v_{m+1}\|^2 - \|v_m\|^2 \right) + a_m \leq C \left\{ \|v_m\|^2 + \|v_{m+1}\|^2 + A_m \right\}$$

hold for

$$m = 0, 1, \ldots, M - 1, \quad \tau = T/M, \quad 0 < T < \infty, \quad a_m \geq 0.$$  

Then positive constants $\tau_0$ and $C$ exist such that for $0 < \tau \leq \tau_0$ and $j = 1, 2, \ldots, M$

$$(2.25) \|v_j\|^2 + \sum_{m=0}^{j-1} \tau a_m \leq C(\|v_0\|^2 + \sum_{m=0}^{j-1} \tau A_m).$$

**Proof.** From (2.24) it follows that

$$(2.26) (1 - C\tau) \|v_{m+1}\|^2 - (1 + C\tau) \|v_m\|^2 + \tau a_m \leq C\tau A_m.$$  

Let us define

$$g(\tau) = \frac{1 - C\tau}{1 + C\tau}.$$  

If $\tau$ is sufficiently small, then

$$(2.27) 0 < g_0 \leq g(\tau)^m \leq g_1 < \infty$$

holds for any $0 \leq m \leq T/\tau$.  

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Multiplying (2.26) by \((1 + C\tau)^{-1} g(\tau)^m\) and using (2.27), we obtain
\[(2.28) \quad g(\tau)^{m+1} \|v_{m+1}\|^2 - g(\tau)^m \|v_m\|^2 + \tau \gamma a_m \leq C_1 \tau A_m,\]
where \(\gamma\) and \(C_1\) are some positive constants, independent of \(m\) and \(\tau\). Let us sum up (2.28) from \(m = 0\) to \(m = j - 1\) to obtain
\[g(\tau)^j \|v_j\|^2 - \|v_0\|^2 + \sum_{m=0}^{j-1} \tau a_m \leq C_1 \sum_{m=0}^{j-1} A_m.\]

Using again (2.27), we arrive at (2.25).

Applying Lemma 4 to
\[\|v_m\| = \|g_m\|^2, \quad \alpha_m = \|g_m\|^2, \quad \gamma_m = \|\tau g_m\|^2 + \psi_m,\]
we derive
\[(2.29) \quad \|g_j\|^2 + \sum_{m=0}^{j-1} \tau \|g_m\|^2 \leq C \left( \|g_0\|^2 + \sum_{m=0}^{j-1} \tau (\|\tau g_m\|^2 + \psi_m) \right),\]
for \(j = 1, 2, \ldots, M\). The last term of (2.29) may be bounded as follows
\[(2.30) \quad \sum_{m=0}^{j-1} \tau (\zeta_m, \delta g_m) = -\tau \sum_{m=1}^{j-1} (\delta \zeta_{m-1}, g_m) - (\tau \zeta_{j-1}, g_j) + (\tau \zeta_0, g_0) \leq \tau \|\zeta_0\| \|g_0\| + \|\tau \zeta_{j-1}\| \|g_j\| + \tau \sum_{m=1}^{j-1} \|\delta \zeta_{m-1}\| \|g_m\| \leq \frac{1}{2} \|\zeta_0\|^2 \|g_0\|^2 + \frac{1}{2} \|\zeta_{j-1}\|^2 \|g_j\|^2 + C_1 \|\tau \zeta_{j-1}\|^2 + \sum_{m=1}^{j-1} \tau (\|\delta g_m\|^2 + C_1 \|\zeta_{m-1}\|^2).
\]

Inserting (2.30) and (2.19) into (2.29), we obtain
\[\|g_j\|^2 (1 - C\epsilon) + \sum_{m=0}^{j-1} \tau \|g_m\|^2 (1 - C\epsilon) \leq C(\|g_0\|^2 + \beta_j),\]
where
\[\beta_j = \|\tau \zeta_0\|^2 + \|\tau \zeta_{j-1}\|^2 + \sum_{m=0}^{j-2} \tau \|\delta \zeta_m\|^2 + \sum_{m=0}^{j-1} \tau \left( \|\zeta_m\|^2 + \|g_m\|^2 + \frac{1}{\tau} \|\eta_m\|^2 \right).\]

Choosing \(\epsilon\) small enough, we shall have
\[(2.31) \quad \|g_j\| \leq C(\|g_0\| + \beta_{M+1}^{1/2})\]
for any \(j = 0, 1, \ldots, M\).
By virtue of (2.7)

\[ \|z_m\| \leq \|\vartheta_m\| + \|\eta_m\|, \quad \|\vartheta_0\| \leq \|z_0\| + \|\eta_0\|. \]

Consequently, (2.31) yields

\begin{equation}
(2.32) \quad \|z_j\| \leq C(\|z_0\| + \|\eta\|_{L^\infty(L_2)} + \|\beta_{M}^{1/2}\|).
\end{equation}

From (1.9) it follows

\[ \|z_0\| \leq \|\varphi - \chi\|, \quad \chi \in \mathcal{M}_h. \]

Since \( \varphi \in H^{2s}, s \geq r \), there exist a constant \( C \) and \( \chi_0 \in \mathcal{M}_h \) such that

\[ \|\varphi - \chi_0\| \leq Ch^{2r}\|\varphi\|_{L^r} \leq Ch^{2r}\|u\|_{W^r}. \]

Hence we obtain

\begin{equation}
(2.33) \quad \|z_0\| \leq Ch^{2r}\|u\|_{W^r}.
\end{equation}

Next let us estimate the terms of \( \beta_M \). We can check easily that

\[ \zeta_m = \int_0^{t/2} P_1(s_m) u^{(4)}(s_m) \, ds_m + \int_{t/2}^t P_2(s_m) u^{(4)}(s_m) \, ds_m, \]

where

\[ u^{(4)} = \frac{\partial^4 u}{\partial t^4}, \quad s_m = t - m\tau, \quad m = 0, 1, \ldots, M - 1, \]

\[ P_1(s) = s^2 \left( -\frac{1}{2} + \frac{2}{3\tau} s \right), \quad P_2(s) = P_1(\tau - s). \]

Consequently

\[ \|\zeta_m\| \leq \int_0^{t/2} |P_1| \|u^{(4)}\| \, ds_m + \int_{t/2}^t |P_2| \|u^{(4)}\| \, ds_m \leq C\tau^3\|u^{(4)}\|_{L^\infty(L_2)}, \]

where \( C \) is independent of \( \tau \) and \( u \),

\begin{equation}
(2.34) \quad \|\zeta_m'\| \leq C\tau^4 \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L^\infty(L_2)} \leq C\tau^4 \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L^\infty(H^2)} \leq C\tau^4\|u\|_{W^5}.
\end{equation}

Similarly,

\[ \delta\zeta_m = \zeta_{m+1} - \zeta_m = \int_0^{t/2} P_1(s_m) u^{(5)}(s_m) \, ds_m + \int_{t/2}^t P_2(s_m) u^{(5)}(s_m) \, ds_m + \]

\[ + \int_t^{3t/2} Q_1(s_m) u^{(5)}(s_m) \, ds_m + \int_{3t/2}^{2t} Q_1(s_m) u^{(5)}(s_m) \, ds_m, \]

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where

\[ u^{(s)} = \frac{\partial^5 u}{\partial t^5}, \quad s_m = t - m\tau, \]

\[ P_1(s) = \frac{1}{6} s^3 \left( -1 + \frac{s}{\tau} \right), \quad P_2(s) = -\frac{1}{12} \left( \frac{\tau}{2} \right)^3 - \frac{1}{6} \left( \frac{\tau}{2} \right)^2 \left( s - \frac{\tau}{2} \right) + \]
\[ + \frac{1}{6} \left( s - \frac{\tau}{2} \right)^3 - \frac{1}{6\tau} \left( s - \frac{\tau}{2} \right)^4, \]

\[ Q_2(s) = P_2(2\tau - s), \quad Q_1(s) = P_1(2\tau - s). \]

Consequently

\[ |P_i(s_m)| \leq C_1 \tau^3, \quad |Q_i(s_m)| \leq C_1 \tau^3, \quad i = 1, 2, \]

\[ \|\delta s_m\| \leq C_2 \tau^3 \int_0^{2\tau} \|u^{(5)}(s_m)\| ds_m \leq 2C_2 \tau^4 \|u^{(5)}\|_{L^\infty(L^2)} \leq C_3 \tau^4 \|u\|_{W^5}, \]

\[ (2.35) \]

\[ \sum_{m=0}^{M-2} \tau \|\delta s_m\|^2 \leq C\tau^8 \|u\|^2_{W^5}. \]

Furthermore,

\[ q_m = \int_0^{\tau/2} P_1(s_m) u^{(5)}(s_m) ds_m + \int_{\tau/2}^{\tau} P_2(s_m) u^{(5)}(s_m) ds_m, \]

where

\[ P_1(s) = \frac{1}{12} s^3 \left( -\frac{1}{3} + \frac{1}{2\tau} s \right), \quad P_2(s) = P_1(\tau - s), \]

so that

\[ |P_i(s)| \leq C\tau^3, \quad i = 1, 2. \]

Consequently

\[ \|q_m\|_{-1} \leq C\tau^3 \int_0^\tau \|u^{(5)}(s_m)\| ds_m \leq C\tau^3 \int_0^\tau \|u^{(5)}(s_m)\| ds_m, \]

\[ (2.36) \]

\[ \sum_{m=0}^{M-1} \tau \|q_m\|^2 \leq \sum_{m=0}^{M-1} C\tau^8 \int_0^\tau \|u^{(5)}(s_m)\|^2 ds_m \leq C\tau^8 \|u^{(5)}\|^2_{L^2(L^2)} \leq C\tau^8 \|u\|^2_{W^5}. \]

Since

\[ \frac{1}{\tau} A\eta_m = \frac{1}{\tau} \left( \eta_m - 2\eta_{m+1/2} + \eta_{m+1} \right) = \frac{1}{\tau} \left( \int_{(m+1/2)\tau}^{(m+1)\tau} \frac{\partial \eta}{\partial t} dt - \int_{m\tau}^{(m+1/2)\tau} \frac{\partial \eta}{\partial t} dt \right), \]

we have

\[ \|\frac{1}{\tau} A\eta_m\|_{-1} \leq \tau^{-1} \left\{ \int_{(m+1/2)\tau}^{(m+1)\tau} \left\| \frac{\partial \eta}{\partial t} \right\|^2 dt + \int_{m\tau}^{(m+1/2)\tau} \left\| \frac{\partial \eta}{\partial t} \right\|^2 dt \right\} = \]
\[ = \tau^{-1} \int_{m\tau}^{(m+1)\tau} \left\| \frac{\partial \eta}{\partial t} \right\|^2 dt. \]
Hence
\[ (2.37) \sum_{m=0}^{M-1} \left| \frac{1}{\tau} \Delta \eta_m \right|^2 \leq \int_0^T \left| \frac{\partial \eta}{\partial t} \right|^2 \, dt = \left| \frac{\partial \eta}{\partial t} \right|_{L^2(H^{-1})}^2. \]

Similarly
\[ \left| \frac{1}{\tau} \delta \eta_m \right|_{L^2(H^{-1})}^2 \leq \tau^{-1} \int_0^{\tau(m+1)} \left| \frac{\partial \eta}{\partial t} \right|_{L^2(H^{-1})}^2 \, dt \]
and therefore
\[ (2.38) \sum_{m=0}^{M-1} \left| \frac{1}{\tau} \delta \eta_m \right|^2 \leq \int_0^T \left| \frac{\partial \eta}{\partial t} \right|^2 \, dt = \left| \frac{\partial \eta}{\partial t} \right|_{L^2(H^{-1})}^2. \]

Finally,
\[ (2.39) \beta_M^2 \leq C \left\{ \tau^4 \left\| u \right\|_{W^s} + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(H^{-1})} \right\}. \]

From (2.32), (2.33), (2.39) and Lemma 3 it follows that
\[ \left\| z \right\| \leq C \left\{ h^{2r} \left\| u \right\|_{W^r} + \left\| \eta \right\|_{W^s} + \tau^4 \left\| u \right\|_{W^s} + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(H^{-1})} \right\} \leq C_3 (h^{2r} \left\| u \right\|_{W^r} + \tau^4 \left\| u \right\|_{W^s}), \quad j = 0, 1, \ldots, M. \]

Note that for \( 0 \leq r \leq s \) and \( u \in G^s \)
\[ \left\| u \right\|_{W^r} \leq \left\| u \right\|_{W^s}. \]

Then Lemma 1 implies
\[ h^{2r} \left\| u \right\|_{W^r} + \tau^4 \left\| u \right\|_{W^s} \leq (h^{2r} + \tau^4) \left\| u \right\|_{W^s} \leq C(h^{2r} + \tau^4) \left\| u \right\|_{G^s} = C(h^{2r} + \tau^4) \left( \left\| \varphi \right\|_{2s} + \left\| f \right\|_{W^{s-1}} \right) \]
and Theorem 2 is proved completely.

References

Souhrn

$L_2$ – ODHADY CHYB SEMI-VARIAČNÍ METODY PRO PARABOLICKÉ ROVNICE

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