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Aplikace matematiky, Vol. 20 (1975), No. 1, 67--72

Persistent URL: <http://dml.cz/dmlcz/103567>

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ALGORITMY

38. CHARPOL

COMPUTATION OF THE CHARACTERISTIC POLYNOMIAL OF MATRICES

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Procedure *CHARPOL* (*b*, *n*, *ep*, *t*, *L*);

comment *b* is a matrix, *n* being its range, *ep* the admissible inaccuracy, *t* is an array of the coefficients of the characteristic polynomial $t_n x^n + t_{n-1} x^{n-1} + \dots + t_0$, *L* is the label of the failure of the method;

integer *n*;

real *ep*;

array *b*, *t*;

label *L*;

begin integer *i*, *j*, *n1*, *k*, *k1*, *k2*, *index*, *r*;

real *s*, *s1*, *maxi*;

array *a* [0 : 15, 0 : 15], *sig* [1 : 15], *p*[0 : 160];

n1 := (*n* + 1) × (*n* + 2)/2; *k2* := 1; *a*[1, 0] := 1;

for *i* := 2 **step** 1 **until** *n* **do** *a*[*i*, 0] := 0;

POKR: **for** *i* := 1 **step** 1 **until** *n* **do**

for *j* := 1 **step** 1 **until** *n* **do**

a[*i*, *j*] := *b*[*i*, *j*];

for *r* := 1 **step** 1 **until** *n* **do**

begin for *i* := 1 **step** 1 **until** *r* **do**

begin *s* := 0; *k1* := *r* - *i*;

for *k* := *r* **step** 1 **until** *n* **do**

s := *s* + *a*[*i*, *k*] × *a*[*k*, *k1*];

k1 := *i* - *i*;

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    if  $k1 < 1$  then go to VYPH;
    for  $k := 1$  step 1 until  $k1$  do
         $s := s - a[i, k - 1] \times a[k, r]$ ;
VYPH: if  $abs(a[i, i - 1]) \leq ep$  then go to VEN;
         $a[i, r] := s/a[i, i - 1]$ 
    end;
     $maxi := 0$ ;  $index := 0$ ;
    if  $r = n$  then go to PIS;
    for  $i := r + 1$  step 1 until  $n$  do
        begin  $s := 0$ ;  $k1 := r - 1$ ;
            for  $k := r$  step 1 until  $n$  do
                 $s := s + a[i, k] \times a[k, k1]$ ;
            for  $k := 1$  step 1 until  $r$  do
                 $s := s - a[i, k - 1] \times a[k, r]$ ;
            VYP:  $a[i, r] := s$ ;
            if  $r = n - 1$  then go to PIS;
             $s1 := abs(s)$ ;
            if  $s1 > maxi$  then begin  $maxi := s1$ ;  $index := i$  end
        end;
         $sig[r] := index$ ;
        if  $index = r + 1$  then go to PIS;
         $k := r + 1$ ;
         $k1 := index$ ;
        for  $i := 0$  step 1 until  $n$  do
            begin  $s := a[k1, i]$ ;  $a[k1, i] := a[k, i]$ ;  $a[k, i] := s$ 
            end;
        for  $i := 1$  step 1 until  $n$  do
            begin  $s := a[i, k1]$ ;  $a[i, k1] := a[i, k]$ ;  $a[i, k] := s$  end;
PIS: end;
    go to HES;
VEN: if  $k2 = n$  then go to L;
         $k2 := k2 + 1$ ;  $a[1, 0] := 1$ ;  $i := 2$ ;

```

POK: if $i = k2$ then $a[i, 0] := 1$ else $a[i, 0] := 0$;
 $i := i + 1$;
 if $i > n$ then go to **POKR**;
 go to **POK**;
comment end of the first step, computation of the characteristic polynomial follows;
HES: $p[0] := 1$; $p[1] := a[1, 1]$; $p[2] := -1$;
for $r := 2$ step 1 until n do
begin $k1 := r \times (r + 1)/2$;
 $k2 := r \times (r - 1)/2$;
 $p[k1 + r] := -p[k2 + r - 1]$;
 $s := 0$;
for $j := 1$ step 1 until $r - 1$ do
 $s := s + a[j, r] \times (-1)^{\uparrow(r-j)} \times p[j \times (j - 1)/2]$;
 $p[k1] := s + a[r, r] \times p[k2]$;
for $k := 1$ step 1 until $r - 1$ do
begin $s := 0$;
if $k + 2 > r$ then go to **TR**;
for $j := k + 1$ step 1 until $r - 1$ do
 $s := s + a[j, r] \times (-1)^{\uparrow(r-j)} \times p[j \times (j - 1)/2 + k]$;
TR: $p[k1 + k] := s - p[k2 + k - 1] + a[r, r] \times p[k2 + k]$
end
end;
 $k1 := n \times (n + 1)/2$;
for $j := n$ step - 1 until 0 do $t[j] := p[k1 + j]$
end;

The above procedure consists of 2 steps. In the first one, direct reduction of the $n \times m$ matrix B to the special Hessenberg form H with unit subdiagonal elements is carried out:

$$(1.1) \quad H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1,n-1} & h_{1n} \\ 1 & h_{22} & h_{23} & \dots & h_{2,n-1} & h_{2n} \\ 0 & 1 & h_{33} & \dots & h_{3,n-1} & h_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & h_{nn} \end{bmatrix}.$$

The elements of the matrix H are computed by columns in n steps according to the formula $B \times M = M \times H$, where

$$M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & m_{22} & 0 & \dots & 0 \\ 0 & m_{32} & m_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & m_{n2} & m_{n3} & \dots & m_{nn} \end{bmatrix}.$$

Thus, the elements of the matrix H and M in the r -th step have respectively the following form:

$$(1.2) \quad h_{ir} = \left(\sum_{k=r}^n b_{ik} \times m_{kr} - \sum_{k=1}^{i-1} m_{ik} \times h_{kr} \right) / m_{ii} \quad \text{for } i \leq r$$

$$(1.3) \quad m_{i,r+1} = \sum_{k=r}^n b_{ik} \times m_{kr} - \sum_{k=1}^r m_{ik} \times h_{kr} \quad \text{for } i > r$$

In order to increase the accuracy of the computation, we find

$$(1.4) \quad \max_{i=r+1}^n |m_{i,r+1}| = |m_{l,r+1}| = \max i.$$

If $l \neq r + 1$ we interchange the rows $(r + 1)$ and l and the columns $(r + 1)$ and l , the same being done in the matrix B and M , H as well. If at any step $\max i = 0$, the computing process is to be carried out from the beginning once more with the new first column of the matrix M :

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{for } n = 5).$$

Therefore it appears obvious that there exist n various trials for reaching a successful solution of the problem given — at the k -th trial, the first column of the matrix M has the first and the k -th components equal to 1; the others are zeros.

In the second step of our algorithm, computation of the characteristic polynomial $t(\lambda) = t_n \lambda^n + t_{n-1} \times \lambda^{n-1} + \dots + t_0$ of the matrix H will be carried out by the recurrent relations [1]:

$$(1.5) \quad \begin{aligned} p_0(\lambda) &= 1 \\ p_1(\lambda) &= h_{11} - \lambda \\ &\dots \dots \dots \\ p_r(\lambda) &= (h_{rr} - \lambda) \times p_{r-1}(\lambda) - h_{r-1,r} \times p_{r-2}(\lambda) + \dots + (-1)^{r-1} \times \\ &\quad \times h_{1r} \times p_0(\lambda), \end{aligned}$$

Then $p_n(\lambda)$ is the characteristic polynomial $t(\lambda)$ of the matrix H and hence also of B . At the same time, h_{ij} in the formula (1.5) are the elements of the matrix H from (1.1).

Furthermore, our procedure is constructed so that only the array B (for the matrix) is required.

The advantages of this method may be seen from the following comparisons with the other three methods:

- a) the method from [1] called briefly WIL in the sequel;
- b) the method KRYLOV from [4];
- c) the method BRYAN from [5];

We have taken seven particular matrices and found the characteristic polynomial of each of them using the methods WIL, KRYLOV, BRYAN and CHARPOL.

	Number of examples solved	Norm of the error	Average time for one case
WIL	2	$0.4 \cdot 10^{-7}$	20"
CHARPOL	7	$0.1 \cdot 10^{-4}$	90"
KRYLOV	6	$0.1 \cdot 10^{-3}$	124.3"
BRYAN	7	$0.1 \cdot 10^{-1}$	49.7"

The first column of this table indicates the number of matrices (from those suggested above) for which the corresponding method worked without failure (yielding the characteristic polynomial). The above table shows that the method WIL which we used to begin with, is quick, though it fails in most cases. Our method, being a little slower one, enables us nevertheless to solve successfully even difficult examples with a satisfactory precision. Now, the method KRYLOV is slow and it also failed in one case. The BRYAN method appears to be quick, however not a sufficiently exact one.

The task of computing the characteristic polynomial of a matrix is often the first step when looking for its eigenvalues. In this case we require the method to be quick and exact at the same time. These requirements have been fulfilled most satisfactorily by our method. Therefore, it may be used for computing eigenvalues of any matrix in connection with the procedure AULEY from [2] or GARSIDE from [3].

Check Example:

$$B = \begin{bmatrix} -2 & 2 & 2 & 2 \\ -3 & 3 & 2 & 2 \\ -2 & 0 & 4 & 2 \\ -1 & 0 & 0 & 5 \end{bmatrix}$$

The characteristic polynomial:

$$t(\lambda) = \lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 24.$$

Literature

- [1] *J. H. Wilkinson*: The algebraic eigenvalue problem. 1965, 357—360, 410—411.
- [2] *Paula Holasová*: Finding zeros of a polynomial by the extended McAuley's method. *Apl. mat.* 17 (1972), 403—407.
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- [4] *Vlado Čech*: Krylov method for calculating the characteristic polynomial of the matrix (in Czech). *Apl. mat.* 18 (1973), 465—468.
- [5] *Petr Škoda, Jiří Neuberg*: Bryan method for calculating the characteristic polynomial of the matrix (in Czech). *Apl. mat.* 18 (1973), 137—139.