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POLYNOMIAL APPROXIMATION AND THE QUADRATURE PROBLEM OVER A SEMI-INFINITE INTERVAL

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INTRODUCTION

The polynomial approximation to a function in a semi-infinite interval is generally obtained by using Laguerre polynomials together with a suitable weight function of the form $w(x) = e^{-x}$. In the 1st part of this paper the authors have obtained a similar expansion of the function $f(x)$ over $(0, \infty)$ in terms of a variant of Chebyshev polynomials of the form $f(x) = \sum_{m=0}^{\infty} a_m T^*_m(e^{-x})$ where $T^*_m(e^{-x}) = \cos m\theta$ with $2e^{-x} - 1 = \cos \theta$, the corresponding weight function being $w(x) = \sqrt{(e^{-x})(1 - e^{-x})^{-1}}$.

In the 2nd part of this paper methods for numerical evaluation of the integral $\int_0^{\infty} e^{-x} f(x) \, dx$ have been developed. The above integral which is usually solved by Laguerre Gauss quadrature method requires the use of Laguerre polynomials. However, in the present method the function $f(x)$ is first expressed in a series of a variant of Chebyshev polynomials as above and then the final evaluation is completed by integrating term by term. Also integrals of the form $\int_{-\infty}^{\infty} e^{-x^2} f(x) \, dx$ which may be reduced to the form $\int_0^{\infty} e^{-x^2} f(x) \, dx$ can be treated similarly. It may be mentioned in this connection that the method for solving the aforesaid integral over $(-\infty, \infty)$ which is evaluated with the help of Hermite polynomials is known as Hermite Gauss quadrature method. Numerical examples have been included to show the practical applications of the present method and to compare and contrast the results with the corresponding Laguerre Gauss and Hermite Gauss methods [1].

POLYNOMIAL APPROXIMATION

Let $f(x)$ be continuous over $(0, \infty)$ and let $T^*_m(e^{-x})$ be a variant of Chebyshev polynomials of degree $m$, where $T^*_m(e^{-x}) = T_m(2e^{-x} - 1) = \cos m\theta$ with $2e^{-x} - 1 = \cos \theta$. 

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Then the Chebyshev-Fourier expansion of $f(x)$ is

\begin{equation}
 f(x) = \sum_{m=0}^{\infty} a_m T_m(e^{-x}), \quad 0 < x < \infty,
\end{equation}

where the prime indicates that the 1st term is to be halved. The polynomials $T_m(e^{-x})$ are orthogonal with respect to the weight function $\omega(x) = \sqrt{[e^{-x})(1 - e^{-x})^{-1}]}$ and we get the following relations

\begin{equation}
 \int_{0}^{\infty} \sqrt{[e^{-x})(1 - e^{-x})^{-1}]} T_m^*(e^{-x}) T_n^*(e^{-x}) \, dx = 0 \quad \text{for} \quad m \neq n,
\end{equation}

$$= \pi \quad \text{for} \quad m = n = 0,$$

$$= \frac{\pi}{2} \quad \text{for} \quad m = n \pm 0.$$

The coefficients $a_m$ of (1) are given by

\begin{equation}
 a_m = \frac{2}{\pi} \int_{0}^{\infty} \sqrt{[e^{-x})(1 - e^{-x})^{-1}]} T_m^*(e^{-x}) f(x) \, dx.
\end{equation}

Assuming that the series (1) has faster rate of convergence an approximation to $f$ may be taken as

\begin{equation}
 f(x) \approx \sum_{k=0}^{N} a_k T_k^*(e^{-x}).
\end{equation}

The coefficients could be calculated from (3) but in practice even for quite simple functions it may be difficult to calculate exactly the integral involved. The approximate computation of the coefficients is done as follows.

The substitution $2e^{-x} = 1 + \cos \theta$ in (3) gives

\begin{equation}
 a_k = \frac{2}{\pi} \int_{0}^{\infty} \cos k\theta f(\log \sec^2 \frac{\theta}{2}) d\theta.
\end{equation}

By using the mid-point quadrature formula in which the abscissae are taken midway between the equidistant points $\theta_i = \pi i/(N + 1)$ gives

\begin{equation}
 a_k \approx a_k = \frac{2}{N + 1} \sum_{i=0}^{N} \cos k\theta_i f(\log \sec^2 \frac{\theta_i}{2})
\end{equation}

where

$$\theta_i = \frac{(2i + 1)\pi}{2(N + 1)}, \quad i = 0, 1, \ldots, N.$$

Thus

\begin{equation}
 a_k \approx a_k = \frac{2}{N + 1} \sum_{i=0}^{N} T_k^*(e^{-x_i}) f(x_i).
\end{equation}
Again substituting this approximate expression for \( a_k \) in (4) we get the polynomial approximation to

\[
\begin{align*}
\text{(8)} & \quad f(x) \approx \sum_{k=0}^{N} \alpha_k T_k^*(e^{-x}) \\
\text{i.e.} & \quad f(x) \approx \sum_{i=0}^{N} \left[ \frac{2}{N+1} \sum_{k=0}^{N} T_k^*(e^{-x}) T_k^*(e^{-x_i}) \right] f(x_i).
\end{align*}
\]

Also

\[
\text{(9)} & \quad 4e^{-x} T_r^*(e^{-x}) = T_{r-1}^*(e^{-x}) + 2T_r^*(e^{-x}) + T_{r+1}^*(e^{-x}).
\]

Putting

\[
\text{(10)} & \quad \psi(x) = \sum_{k=0}^{N} T_k^*(e^{-x_i}) T_k^*(e^{-x})
\]

and employing (9) we obtain

\[
\text{(11)} & \quad 4e^{-x} \psi(x) = \sum_{k=0}^{N} 4e^{-x} T_k^*(e^{-x}) T_k^*(e^{-x_i}) = \\
& \quad = 2e^{-x} + \sum_{k=1}^{N} \left[ T_{k+1}^*(e^{-x}) + 2T_k^*(e^{-x}) + T_{k-1}^*(e^{-x}) \right] T_k^*(e^{-x_i})
\]

and

\[
\text{(12)} & \quad 4e^{-x_i} \psi(x) = 2e^{-x_i} + \sum_{k=1}^{N} \left[ T_{k+1}^*(e^{-x_i}) + 2T_k^*(e^{-x_i}) + T_{k-1}^*(e^{-x_i}) \right] T_k^*(e^{-x}).
\]

Now subtracting (12) from (11) we get

\[
\text{(13)} & \quad \psi(x) = \frac{T_{N+1}^*(e^{-x}) T_N^*(e^{-x_i})}{4(e^{-x} - e^{-x_i})}.
\]

Again

\[
\text{(14)} & \quad e^{x_i} T_{N+1}^*(e^{-x_i}) T_N^*(e^{-x_i}) = -2(N+1).
\]

Hence from (8), (10), (13) and (14) we obtain

\[
\text{(15)} & \quad f(x) \approx \sum_{i=0}^{N} \left[ \frac{T_{N+1}^*(e^{-x})}{\{1 - e^{-(x-x_i)}\}} \frac{T_N^*(e^{-x_i})}{T_{N+1}^*(e^{-x_i})} \right] f(x_i).
\]

QUADRATURE PROBLEM

The evaluation of the integral \( \int_{0}^{\infty} e^{-x} f(x) \, dx \) can be done in two ways. In the first case the function \( f(x) \) is replaced by the expression contained in (4), whence we get

\[
\text{(16)} & \quad \int_{0}^{\infty} e^{-x} f(x) \, dx \approx \sum_{k=0}^{N} \alpha_k \int_{0}^{\infty} e^{-x} T_k^*(e^{-x}) \, dx = \sum_{p=0}^{[N/2]} a_{2p} \frac{a_{2p,0}}{1 - 4p^2},
\]

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where \([N/2]\) means the largest integer contained in \(N/2\) for a given \(N\), the coefficients \(a_k\) being calculated from (7).

In the other case we replace \(f(x)\) by (15) so that

\[
\int_0^\infty e^{-x} f(x) \, dx \approx \sum_{i=0}^{N} \frac{f(x_i)}{e^{x_i} T_{N+1}^{*}(e^{-x_i})} \int_0^\infty e^{-x} T_{N+1}^{*}(e^{-x}) \, dx.
\]

Applying (10) and (13), (17) reduces to

\[
\int_0^\infty e^{-x} f(x) \, dx \approx \sum_{i=0}^{N} \left[ \frac{2}{N+1} \sum_{k=0}^{N} T_k^{*}(e^{-x_i}) \int_0^\infty e^{-x} T_k^{*}(e^{-x}) \, dx \right] f(x_i) = \sum_{i=0}^{N} C_i f(x_i),
\]

where

\[
C_i = \frac{2}{N+1} \sum_{p=0}^{[N/2]} T_2^p(e^{-x_i}).
\]

The same result is obtained if the function \(f(x)\) in the previous integral is replaced by (8).

**NUMERICAL EXAMPLES**

We consider the following numerical examples:

(a) \(I = \int_0^\infty \frac{x \, dx}{1 - e^{-2x}} = 1.2337005\),

(b) \(I = \int_0^\infty e^{-x} \sin x \, dx = 0.5\),

(c) \(I = \int_{-\infty}^\infty e^{-x^2} \cos x \, dx = 1.3803884\).

The numerical details of the above examples are contained in table 1, 2 and 3 respectively.

**Remarks**

(i) It may be seen from the above tables that to achieve the desired accuracy in some case larger number of points are required to evaluate the integral in the present method than in the corresponding Laguerre-Gauss quadrature and Hermite-Gauss quadrature methods. This is the only drawback of this method. But owing to the easy availability of a computer now-a-days such a defect should not be taken into account.
so seriously because it involves only a little more computing time in comparison to other methods. On the other hand the existing methods require the use of precomputed weight coefficients and the abscissae which should be known in advance, either in the form of a table. But no such previous data are required in the present method which is the advantage of it.

(ii) In the evaluation of the integral the formula (18) should be preferred to formula (16) because the weight coefficients $C_i$ in (18) can be calculated beforehand from (19) for specified values of $N$ and can be supplied in the form of a table. This saves a lot of computing time for a particular evaluation of an integral.

(iii) It appears from the above tables that although in some cases larger number of points are required in the present method as compared to Laguerre-Gauss of Laguerre-Hermite methods, as the case may be, the results obtained by the present method are.
method deviate less from the actual values than those of other methods. Thus by taking a few more points more accuracy in the solution is achieved.

(iv) No attempts have been made to obtain the error estimates both for the polynomial approximation and the integral evaluation. But simple estimates in these cases, if necessary, can be easily obtained by the methods given in [2].

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References


Souhrah

APROXIMACE POLYNOMY A PROBLÉM KVADRATURE
NA POLONEKONEČNÉM INTERVALU

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V článku je vypracován způsob aproximace funkce na polonekonečném intervalu \((0, \infty)\) polynomy, při čemž je užita jistá modifikace Čebyševových polynomů. Metoda je aplikována na problém kvadratury na tomtéž intervalu.

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