ON A PROBABILITY INEQUALITY FOR MULTIVARIATE NORMAL DISTRIBUTION

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1. Introduction. Let P_λ denote the p-variate normal distribution $N_p(\mu, \Sigma_\lambda)$, where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \begin{matrix} p_1 \\ p_2 \end{matrix}, \quad \Sigma_\lambda = \begin{bmatrix} \Sigma_{11} & \lambda\Sigma_{12} \\ \lambda\Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{matrix} p_1 \\ p_2 \end{matrix}, \qquad (1)$$
$$\qquad\qquad\qquad p_1 \quad\; p_2$$

$p_1 + p_2 = p$, $0 \leqq \lambda \leqq 1$, and Σ_1 is positive-definite. Let $C_1 \subset R^{p_1}$, $C_2 \subset R^{p_2}$ be convex symmetric (about the respective origins) sets. Define

$$\pi(\lambda) = P_\lambda[C_1 \times C_2]\,. \qquad (2)$$

Das Gupta et al. [1] have shown that

$$\pi(0) \leqq \pi(1) \qquad (3)$$

under the following assumptions: There exist vectors $b_1 \in R^{p_1}$, $b_2 \in R^{p_2}$ and a scalar c such that

(i) $\mu_i = cb_i$, $i = 1, 2$

(ii) $\Sigma_{12} = b_1 b_2'$

(iii) $\Sigma_{ii} - b_i b_i'$ $(i = 1, 2)$ is positive definite.

The inequality (3) was proved by Khatri [2] when $\mu = 0$. In this note, we shall show that

$$\pi(\lambda) \leqq \pi(\lambda^*) \qquad (4)$$

for $0 \leqq \lambda < \lambda^* \leqq 1$ when the above assumptions (i)–(iii) hold. For motivations and applications of the inequalities (3) and (4), one may see Das Gupta et al. [1]

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and Khatri [2]. The inequality (4) was proved by Šidák [3] under the following stronger assumptions:

(a) $\mu = 0$

(b) $R_{ii} = b_i b_i' + \text{diag}[I - b_i b_i'], i = 1, 2$

$R_{12} = b_1 b_2'$

where $b_1 : p_1 \times 1$, $b_2 : p_2 \times 1$,

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

is the correlation matrix corresponding to Σ_1, and for a square matrix A, diag (A) is defined to be the diagonal matrix obtained from A by replacing all the off-diagonal elements of A by 0.

Our proof essentially uses the inequality (3) and some suitable prior distributions of the parameters. It can also be seen that Šidák's [3] proof may be modified, incorporating the assumptions in this note, to obtain (4).

2. Proof of the inequality (4). Consider $b_1 : p_1 \times 1$, $b_2 : p_2 \times 1$ and c satisfying the assumptions (i)–(iii). Let $X_1 : p_1 \times 1$, $X_2 : p_2 \times 1$ and θ be distributed as the $(p + 1)$-variate normal distribution, such that conditional joint distribution of X_1 and X_2, given θ, is

$$N_p\left[\begin{pmatrix} \theta\, b_1 \\ \theta\, b_2 \end{pmatrix}, \Gamma\right]$$

and $\theta \sim N(c, \lambda)$, ($\lambda$ being the variance of θ). Then the unconditional joint distribution of X_1 and X_2 is

$$N_p\left[\begin{pmatrix} c\, b_1 \\ c\, b_2 \end{pmatrix}, \quad \Gamma + \lambda \begin{pmatrix} b_1 b_1' & b_1 b_2' \\ b_2 b_1' & b_2 b_2' \end{pmatrix}\right].$$

It can be seen that the joint distribution of X_1 and X_2 id P_{λ^*} or P_λ, according as

$$\Gamma = \Gamma_1 \equiv \begin{pmatrix} \Sigma_{11} - \lambda b_1 b_1' & (\lambda^* - \lambda)\, b_1 b_2' \\ (\lambda^* - \lambda)\, b_2 b_1' & \Sigma_{22} - \lambda b_2 b_2' \end{pmatrix}.$$

$$\Gamma = \Gamma_0 \equiv \begin{pmatrix} \Sigma_{11} - \lambda b_1 b_1' & 0 \\ 0 & \Sigma_{22} - \lambda b_2 b_2' \end{pmatrix},$$

where $0 \leqq \lambda < \lambda^* \leqq 1$. Applying the inequality (3) of Das Gupta et. al. [1] after verifying their assumptions, we get

$$(5) \qquad P[X_1 \in C_1,\ X_2 \in C_2 \mid \theta,\ \Gamma = \Gamma_1] \geqq P[X_1 \in C_1,\ X_2 \in C_2 \mid \theta,\ \Gamma = \Gamma_0].$$

Taking expectations of both sides of the above inequality (5) with respect to θ, we get

$$\pi(\lambda^*) \geqq \pi(\lambda).$$

Note 1. If $\mu_1 = 0$, $\mu_2 = 0$, rank $(\Sigma_{12}) = 1$, there exist vectors b_1, b_2 satisfying (ii) and (iii). To satisfy (i), take $c = 0$. To see this, note that there exist nonsingular matrices A_1 and A_2 such that

$$A_1\Sigma_{11}A_1' = I_{p_1}, \quad A_2\Sigma_{22}A_2' = I_{p_2},$$

and

$$A_1\Sigma_{12}A_2' = \begin{pmatrix} \varrho & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} : p_1 \times p_2,$$

where $0 \leqq \varrho < 1$.

Define

$$b_i = A_i^{-1}[\sqrt{(\varrho)}\, 0 \dots 0]' : p_i \times 1, \ i = 1, 2.$$

Note 2. Suppose $\mu_1 \neq 0$, $\mu_2 \neq 0$. Assume the following: There exists a positive scalar k such that

(ii') $\Sigma_{12} = k\mu_1\mu_2'$

(iii') $k^{-1} > \max(\mu_1'\Sigma_{11}^{-1}\mu_1, \mu_2'\Sigma_{22}^{-1}\mu_2)$.

We shall show that there exist b_1, b_2 and c satisfying (i)–(iii). There exist orthogonal matrices L_1 and L_2 such that

$$\mu_i' = (\delta_i\, 0 \dots 0)\, L_i\Sigma_{ii}^{1/2}, \quad i = 1, 2$$

where

$$\delta_i = (\mu_i'\Sigma_{ii}^{-1}\mu_i)^{1/2}.$$

Define

$$c = k^{-1/2}, \quad b_i = \mu_i/c, \quad i = 1, 2.$$

Note 3. When Σ_{ii} $(i = 1, 2)$ is p. d., rank $(\Sigma_{12}) = 1$, but Σ is p.s.d., the above proof is also valid for showing

$$\pi(\lambda) \leqq \pi(\lambda^*),$$

where $0 \leqq \lambda < \lambda^* < 1$. In that case we need the following assumption:

(iiia) $$\Sigma_{ii} - \lambda^* b_i b_i' \quad (i = 1, 2,) \text{ is p.d.}$$

instead of the assumption (iii). Correspondingly the assumption (iii') in Note 2 can be changed. However the proof is no longer tenable for showing $\pi(\lambda) \leqq \pi(1)$, $0 \leqq \lambda < 1$ when Σ is p.s.d. subject to the assumptions made in the beginning of Note 3. This may apparently follow from the result of Das Gupta et. al. [1] who claimed to prove (3) under the assumption: $\Sigma_{ii} - b_i b_i'$ is p.s.d. $(i = 1, 2)$, instead of (iii); however their proof is not complete.

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References

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Souhrn

O JEDNÉ NEROVNOSTI PRO PRAVDĚPODOBNOSTI V MNOHOROZMĚRNÉM NORMÁLNÍM ROZLOŽENÍ

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Nechť P_λ označuje p-rozměrné normální rozložení $N_p(\mu, \Sigma_\lambda)$, kde

$$\Sigma_\lambda = \begin{bmatrix} \Sigma_{11} & \lambda\Sigma_{12} \\ \lambda\Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

je rozdělena na bloky s p_1, p_2 řádky a sloupci, přičemž $p_1 + p_2 = p$, $0 \leqq \lambda \leqq 1$, a Σ_1 je positivně definitní. Buďtež $C_1 \subset R^{p_1}$, $C_2 \subset R^{p_2}$ konvexní symetrické množiny. V článku je za určitých předpokladů o μ a Σ_1 dokázáno, že pro $0 \leqq \lambda < \lambda^* \leqq 1$ je $P_\lambda[C_1 \times C_2] \leqq P_{\lambda^*}[C_1 \times C_2]$, což zobecňuje dřívější výsledky Das Gupty aj. [1], Khatriho [2] a Šidáka [3].

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