Jaroslav Haslinger; Ivan Hlaváček
Convergence of a finite element method based on the dual variational formulation

Aplikace matematiky, Vol. 21 (1976), No. 1, 43--65

Persistent URL: http://dml.cz/dmlcz/103621
CONVERGENCE OF A FINITE ELEMENT METHOD BASED ON THE DUAL VARIATIONAL FORMULATION

JAROSLAV HASLINGER and IVAN HLAVÁČEK

(Received February 7, 1975)

INTRODUCTION

In practice, one often meets problems, when the cogradient of the solution is more important than the solution itself (e.g. in elastostatics, the stresses are more interesting than the displacements). There are three possibilities how to calculate the cogradient:

(i) an indirect method, based on the minimum potential energy principle. We have to find the solution and then to evaluate its cogradient (Compatible models);

(ii) to apply the dual variational formulation (principle of minimum complementary energy), which yields the cogradient directly (Equilibrium models);

(iii) to apply a mixed variational principle (of the Hellinger-Reissner type), enabling to get both the solution and its cogradient simultaneously (Hybrid models).

In the present paper, we study a particular procedure of the class (ii), starting from a dual variational formulation. A number of articles has been written on the dual finite element analysis in elastostatics (see e.g. the works [1], [2] by B. Fraeijs de Veubeke or V. B. Watwood, Jr., B. J. Hartz [3]) and in general potential problems (B. Fraeijs de Veubeke, M. Hogge [4]). To the authors' knowledge, however, no theoretical convergence result has been presented for the problems mentioned above. This was done only for a special class of problems by J. P. Aubin, H. G. Burchard [5] and J. Vacek [6], who studied the equations with an “absolute term” (i.e. \( A(u) + a_0u = f, \ a_0 \neq 0 \)). The dual (also “conjugate”) method for equations of this type can be extended to parabolic problems, where the role of the absolute term \( a_0u \) is played by the time-derivative \( \frac{\partial u}{\partial t} \) (see I. Hlaváček [7], H. Gajewski [13]).

We concentrated our effort to the convergence analysis of a simplest finite element "equilibrium model", applying the piecewise linear polynomials to the solution of a mixed boundary value problem for one second order elliptic equation without the absolute term.
In Section 1 we employ the method of orthogonal projection to derive the principle of minimum complementary energy in a “weak form” and its finite-dimensional analogue. In Section 2 a subspace of triangular finite elements is constructed, which consists of piecewise linear vector-functions such that their divergence vanishes in the whole domain under consideration in the sense of distributions. The subspace is applied to the dual variational method in Section 3 with special regard to the approximation of an inhomogeneous natural boundary condition. As a result, the procedure is shown to be second order correct in $h$ (the maximal side of all triangles), provided the exact solution is sufficiently smooth. The last Section 4 contains some a posteriori estimates, following from the use of both the primal and the dual finite element method (cf. also F. Grenacher [8], W. Prager, J. L. Synge [9]).

The authors will present analogous results for $n$-dimensional domains elsewhere (see [12]).

1. SETTING OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral domain. Consider a differential operator

$$\mathcal{A}u = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right),$$

where

$$a_{ij} \in L^\infty(\Omega), \quad a_{ij}(x) = a_{ji}(x), \quad i, j = 1, \ldots, n,$$

$$\exists \alpha = \text{const} > 0,$$

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha \|\xi\|^2 \quad \forall \xi \in E_n$$

holds almost everywhere in $\Omega$.

Suppose that the boundary $\Gamma$ of $\Omega$ consists of two disjoint sets $\Gamma_u, \Gamma_g$, which are either empty or open in $\Gamma$ and a set $\mathcal{R}$ of zero $(n-1)$-dimensional measure

$$\Gamma = \Gamma_u \cup \Gamma_g \cup \mathcal{R}, \quad \Gamma_u \cap \Gamma_g = \emptyset, \quad \text{mes}_{n-1} \mathcal{R} = 0.$$

Moreover, let $\Gamma_u$ be non-empty.\(^2\)

We shall solve the following problem

$$\begin{align*}
\mathcal{A}u &= f \quad \text{in} \ \Omega \\
u &= \bar{u} \quad \text{on} \ \Gamma_u \\
\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} n_i &= g \quad \text{on} \ \Gamma_g,
\end{align*}$$

\(^1\) I.e. $a_{ij}$ are bounded measurable functions on $\Omega$.

\(^2\) If $\Gamma_u$ is empty, we have to introduce subspaces $V_p \subset V$ of “normalized” test functions to obtain uniqueness of solution and the generalized Friedrichs inequality.
where \( f \in L^2(\Omega) \), \( \bar{u} \in W^{1,2}(\Omega) \) and \( g \in L^2(\Gamma_g) \) are given functions, \( n \), being the components of the unit outward normal to \( \Gamma \).

Let us denote

\[
V = \{ v \mid v \in W^{1,2}(\Omega) , \ v = 0 \text{ on } \Gamma_u \} .
\]

A function \( u \in W^{1,2}(\Omega) \) will be called a weak solution of the problem (1.3) if

\[
(1.4) \quad u - \bar{u} \in V ,
\]

\[
a(u,v) = \int_\Omega f v \, dx + \int_{\Gamma_g} g v \, d\Gamma , \ \forall v \in V ,
\]

where

\[
a(u, v) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx .
\]

From (1.1), (1.2) and the Lax-Milgram theorem the existence and uniqueness of \( u \) follows. Moreover, it holds

\[
(1.5) \quad \| u \|_{W^{1,2}(\Omega)} \leq C(\| f \|_{L^2(\Omega)} + \| \bar{u} \|_{W^{1,2}(\Omega)} + \| g \|_{L^2(\Gamma_g)}) .
\]

Introducing the functional of potential energy

\[
\mathcal{L}(v) = \frac{1}{2} a(v, v) - \int_\Omega f v \, dx - \int_{\Gamma_g} g v \, d\Gamma ,
\]

we may formulate the minimum potential energy principle

\[
\mathcal{L}(u) = \min_{v \in \bar{u} + V} \mathcal{L}(v)
\]

where \( \bar{u} + V \) denotes the set of all sums \( \bar{u} + v, v \in V \).

Let \( H = [L^2(\Omega)]^n \) with the norm

\[
\| \lambda \|^2 = \sum_{i=1}^n \| \lambda_i \|^2 = \sum_{i=1}^n \int_\Omega \lambda_i^2 \, dx , \ \lambda = (\lambda_1, \ldots, \lambda_n)
\]

Introduce a bilinear form

\[
(1.6) \quad (\lambda, \mu)_H = \sum_{i,j=1}^n \int_\Omega b_{ij}(x) \lambda_i \mu_j \, dx , \ \lambda, \mu \in H ,
\]

where \( b_{ij} \) are the entries of the matrix \( [a^{-1}] \) inverse to \( [a] \).

Making use of (1.1), (1.2), we can see that (1.6) defines a scalar product on \( H \), and such constants \( c_1 > 0, \ c_2 > 0 \) exist that

\[
(1.6') \quad c_2 \| \lambda \| \leq \| \lambda \|_H \leq c_1 \| \lambda \| \ \forall \lambda \in H ,
\]
where
\[\|\lambda\|_H^2 = (\lambda, \lambda)_H.\]

Hence the space \( H \) with the scalar product (1.6) is a Hilbert space.

Let us define
\[H_t = \{ \lambda | \lambda \in H, \exists v \in V : \lambda_i = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j} \},\]
\[H_2 = \{ \lambda | \lambda \in H, B(\lambda, v) = 0 \ \forall v \in V \},\]

(1.7)
\[B(\lambda, v) = \sum_{i=1}^{n} \lambda_i \frac{\partial v}{\partial x_i} \ dx.\]

**Lemma 1.1.**

1° \( H_1 \) and \( H_2 \) are (closed) subspaces of \( H \);
2° \( H_1 \perp H_2 \);
3° \( H = H_1 \oplus H_2 \)

**Proof**

1° Let \( \lambda^m \in H_1, \lambda^m \to \lambda \) in \( H \). Then we may write
\[\lambda^m_i = \sum_{j=1}^{n} a_{ij} \frac{\partial v_m}{\partial x_j}, \ v_m \in V,\]

(1.8)
\[\|\lambda^m - \lambda^p\|_H^2 = \int_{\Omega} \sum_{i,j=1}^{n} b_{ij}(\lambda^m_i - \lambda^p_i)(\lambda^m_j - \lambda^p_j) \ dx = \]
\[= \int_{\Omega} \sum_{i,j=1}^{n} b_{ij} \left( \sum_{i=1}^{n} a_{ii} \frac{\partial}{\partial x_i} (v_m - v_p) \right) \left( \sum_{k=1}^{n} a_{kk} \frac{\partial}{\partial x_k} (v_m - v_p) \right) \ dx \geq \]
\[\geq \alpha \int_{\Omega} |\text{grad} (v_m - v_p)|^2 \ dx \geq c \|v_m - v_p\|_{W^{2,2}(\Omega)}^2.\]

where the definition of \([b_{ij}]\), (1.2) and the generalized Friedrichs inequality for \( v \in V \) (see [10]) have been used. Hence \( v_n \to v \) in \( V \),
\[\lambda_i = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j}, \ \lambda \in H_1.\]

By virtue of (1.6) and the definition (1.7), the set \( H_2 \) is closed. The linearity is obvious.

2° Let \( \lambda^' \in H_1, \lambda^'' \in H_2 \). Then we have
\[(\lambda^', \lambda^'')_H = \int_{\Omega} \sum_{i,j=1}^{n} b_{ij} \left( \sum_{k=1}^{n} a_{ik} \frac{\partial v}{\partial x_k} \right) \lambda^''_j \ dx = B(\lambda^'', v) = 0,\]

using the definition of \([b_{ij}]\) and \( H_2 \).
3° Let $H^1_t$ denote the orthogonal complement of $H_t$ in $H$. From the point 2° it follows that $H_2 \subset H^1_t$. Conversely, let $z \in H^1_t$. Then
\[
\int_{\Omega} \sum_{i,j=1}^{n} b_{ij} \left( \sum_{k=1}^{n} a_{ik} \frac{\partial u}{\partial x_k} \right) z_j \, dx = 0 \quad \forall v \in V.
\]

The properties of $[b_{ij}]$ yield that
\[
\int_{\Omega} \sum_{j=1}^{n} \frac{\partial v}{\partial x_j} z_j \, dx = 0 \quad \forall v \in V,
\]
Hence $z \in H_2$. Q.E.D.

Let us introduce
\[
(1.9) \quad \lambda_i(v) = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j}, \quad \lambda(v) = (\lambda_1(v), \ldots, \lambda_n(v)), \quad v \in W^{1,2}(\Omega)
\]
\[
A_{f,g} = \left\{ \lambda \mid \lambda \in H, B(\lambda, v) = \int_{\Omega} f v \, dx + \int_{\Gamma_g} g v \, d\Gamma, \forall v \in V \right\}.
\]

The dual variational formulation is the following

**Theorem 1.1. (Principle of minimum complementary energy).**

Let $B(\lambda, v)$ and $\lambda(v)$ be defined through (1.7) and (1.9), respectively. Let $u \in W^{1,2}(\Omega)$ be the weak solution of the problem (1.3). Then the functional
\[
\mathcal{J}(\lambda) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} b_{ij} \lambda_i \lambda_j \, dx - B(\lambda, \bar{u})
\]
attains its minimum on the set $A_{f,g}$, if and only if $\lambda = \lambda(u)$.\(^{*}\)

**Proof.** (cf. [11]). Let us set $u = \bar{u} + w, w \in V$. Then $\lambda(w) \in H_t$ and $\lambda - \lambda(u) \in H_2$ for all $\lambda \in A_{f,g}$.

The functional
\[
\mathcal{J}(\lambda) = \| \lambda - \lambda(\bar{u}) \|^2_H = \| (\lambda - \lambda(u)) + (\lambda(u) - \lambda(\bar{u})) \|^2_H =
\]
\[
= \| \lambda - \lambda(u) \|^2_H + \| \lambda(w) \|^2_H
\]
attains its minimum on $A_{f,g}$, iff $\lambda = \lambda(u)$.

Rearranging leads to the formula
\[
\mathcal{J}(\lambda) - \| \lambda(\bar{u}) \|^2_H = \| \lambda \|^2_H - 2 \lambda, \lambda(\bar{u})_H =
\]
\[
= \int_{\Omega} \sum_{i,j=1}^{n} b_{ij} \lambda_i \lambda_j \, dx - 2B(\lambda, \bar{u}) = 2\mathcal{J}(\lambda),
\]
which yields the assertion of the theorem. Q.E.D.

\(^{*}\) It holds that $-\mathcal{J}(\lambda(u)) = \mathcal{J}(u) + \int_{\Omega} f \bar{u} \, dx + \int_{\Gamma_g} g \bar{u} \, d\Gamma$.

47
We obtained a variational problem of a minimum of a quadratic functional on a closed convex set $A_{f,g} \subset H$. As usual, the problem can be transformed into a similar one, but on a linear space $A_{0,0} = H_2$. Let $\lambda \in A_{f,g}$ be a fixed vector-function. Then

$$A_{f,g} = \lambda + A_{0,0} = \lambda + H_2,$$

i.e., every $\lambda \in A_{f,g}$ can be written in the form $\lambda = \lambda + \chi, \chi \in H_2$. Then

$$\mathcal{J}(\lambda) = \mathcal{J}(\lambda + \chi) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} b_{ij} \lambda_{i} \lambda_{j} \, dx + \int_{\Omega} \sum_{i,j=1}^{n} b_{ij} \lambda_{i} \lambda_{j} \, dx - B(\chi, \bar{u}) + N(\lambda, \bar{u}),$$

where $N(\lambda, \bar{u})$ depends only on $\lambda, \bar{u}$. Let us set

$$\Phi(\lambda) = \frac{1}{2}(\chi, \chi)_H - F(\chi),$$

where

$$F(\chi) = - \int_{\Omega} \sum_{i,j=1}^{n} b_{ij} \lambda_{i} \lambda_{j} \, dx + B(\chi, \bar{u}).$$

Then we may replace the minimum problem of Theorem 1.1 for $\mathcal{J}(\lambda)$ by an equivalent problem:

to find $\chi^0 \in H_2$ such that

$$\Phi(\chi^0) = \min_{\chi \in H_2} \Phi(\chi).$$

Let $h \in (0, 1)$ and let $\{V_h\}$ be a system of finite-dimensional subspaces of $H_2$. We define the following procedure:

to find $\chi^0_h \in V_h$ such that

$$\Phi(\chi^0_h) = \min_{\chi \in V_h} \Phi(\chi).$$

**Theorem 1.2.** To every $h \in (0, 1)$ there exists precisely one $\chi^0_h \in V_h$ satisfying (1.12). It holds

$$\|\chi^0 - \chi^0_h\|_H \leq \inf_{\chi \in V_h} \|\chi^0 - \chi\|_H \leq c_1 \inf_{\chi \in V_h} \|\chi^0 - \chi\|$$

**Proof.** The element $\chi^0_h$, satisfying (1.12), is characterized through the relation

$$D\Phi(\chi^0_h, \chi) = 0 \quad \forall \chi \in V_h,$$

where $D\Phi(\chi^0_h, \chi)$ denotes the value of Gâteaux differential of $\Phi$ at the “point” $\chi^0_h$ and in the “direction” $\chi$. We have

$$D\Phi(\chi^0_h, \chi) = (\chi^0_h, \chi)_H - F(\chi),$$

48
consequently
\[(1.14) \quad (\chi^0, \chi)_H = F(\chi) \quad \forall \chi \in V_h.\]
It is readily seen that (1.14) is equivalent with a system of linear equations with a positive definite matrix. Therefore precisely one \(\chi^0_h \in V_h\) exists, satisfying (1.14).

Similarly, by virtue of (1.11), it holds
\[(1.15) \quad (\chi^0, \chi)_H = F(\chi) \quad \forall \chi \in H_2.\]
Subtracting (1.14) from (1.15) yields that
\[(\chi^0 - \chi^0_h, \chi)_H = 0 \quad \forall \chi \in V_h.\]
Next let \(\chi \in V_h\) be an arbitrary element. Then we may write
\[
\|\chi^0 - \chi^0_h\|^2_H = (\chi^0 - \chi^0_h, \chi^0 - \chi)_H + (\chi^0 - \chi^0_h, \chi - \chi^0)_H = \]
\[
= (\chi^0 - \chi^0_h, \chi^0 - \chi)_H \leq \|\chi^0 - \chi^0_h\|_H \|\chi^0 - \chi\|_H
\]
Dividing by \(\|\chi^0 - \chi^0_h\|_H\) and taking the infimum of the right-hand side leads to (1.13), if we use also (1.6'). Q.E.D.

2. THE SUBSPACES \(\mathcal{N}_h\)

We shall restrict ourselves to the plane case \((n = 2)\). Let \(K\) be a triangle with vertices \(a_1, a_2, a_3\) and set \(a_4 = a_1\). We introduce the following notation:
\[
\mathcal{W} = \left[ W^{1,2}(K) \right]^2 = \{ u = (u_1, u_2), \quad u_i \in W^{1,2}(K), \quad i = 1,2 \},
\]
\[
\mathcal{C} = \left[ C(K) \right]^2 \quad \text{with the norm}
\|
\]

\[
\mathcal{C}^2(K) = \left[ C^2(K) \right]^2 \quad \text{with the seminorm}
\]
\[
\| u \|_{\mathcal{C}^2(K)} = \max_{i,j,k=1,2} \left| \frac{\partial^2 u_i(x)}{\partial x_j \partial x_k} \right|
\]

\(n\) is the outward unit normal to \(\partial K\); thus \(n = n(x, y) = n^{(i)} \in R_2\) is constant on the side \(a_ia_{i+1}\), \((i = 1, 2, 3)\) of the triangle \(K\).

For \(u \in \mathcal{W}\) we define the outward flux
\[(2.1) \quad T_i u = u|_{a_i^{(i)}} \cdot n^{(i)} = \bar{u}^{(i)} + \bar{n}^{(i)}, \quad n^{(i)} = (n_1^{(i)}, n_2^{(i)}),
\]
where \(\bar{u}_j\) are the traces of \(u_j\) on \(a_ia_{i+1}\).
By $P_k(M)$, with $k$ a non-negative integer, we denote the set of polynomials of the order at most $k$, defined on the set $M \subset \mathbb{R}_2$.

**Definition 2.1.** We say that $\lambda_1^{(i)}, \lambda_2^{(i)}, (i = 1, 2, 3)$, are basic linear functions of the side $a_i a_{i+1}$, if

$$
\lambda_k^{(i)} \in P_k(a_i a_{i+1}), \quad (k = 1, 2)
$$

$$
\lambda_1^{(i)}(a_i) = 1, \quad \lambda_1^{(i)}(a_{i+1}) = 0.
$$

$$
\lambda_2^{(i)}(a_i) = 0, \quad \lambda_2^{(i)}(a_{i+1}) = 1.
$$

From the definition it follows immediately that

$$
\lambda_1^{(i)} + \lambda_2^{(i)} = 1 \text{ on } a_i a_{i+1},
$$

$$
\int_{a_i a_{i+1}} \lambda_k^{(i)} \, ds = \frac{1}{2} l_i, \quad k = 1, 2,
$$

where $l_i$ denotes the length of $a_i a_{i+1}$.

Henceforth we shall use the notation

$$
\int_{a_i a_{i+1}} u \cdot v \, d\Gamma = [u, v], \quad u, v \in L_2(a_i a_{i+1})
$$

**Lemma 2.1.** Let $\gamma_i, \delta_i \in \mathbb{R}_1, (i = 1, 2, 3)$ be given. Then precisely one $u \in [P_1(K)]^2$ exists such that

$$
T_i u(a_i) = \gamma_i, \quad T_i u(a_{i+1}) = \delta_i, \quad (i = 1, 2, 3).
$$

**Proof.** Denote $u = (u_1, u_2), u_i \in P_1(K)$. It is sufficient to prove that the six values $\{u_j(a_i)\}_{j=1}^3, j = 1, 2$ are determined uniquely by (2.4). Let e.g. $i = 1$. Then

$$
T_1 u(a_1) = u(a_1) \cdot n^{(1)} = \gamma_1
$$

$$
T_3 u(a_1) = u(a_1) \cdot n^{(3)} = \delta_3.
$$

As $n^{(1)}, n^{(3)}$ are linearly independent vectors, $u(a_1)$ is uniquely determined. The same holds at the remaining vertices. Q.E.D.

**Theorem 2.1.** Let $u \in \mathcal{W}$. Then the equations

$$
[T_i u, \lambda_k^{(i)}] = \alpha_i [\lambda_1^{(i)}, \lambda_k^{(i)}] + \beta_i [\lambda_2^{(i)}, \lambda_k^{(i)}], \quad k = 1, 2
$$

$$
\Pi u(a_i) \cdot n^{(i)} = \alpha_i
$$

$$
\Pi u(a_{i+1}) \cdot n^{(i)} = \beta_i,
$$

with $i = 1, 2, 3$, define an operator $\Pi \in \mathcal{L}(\mathcal{W}; [P_1(K)]^2 \cap \mathcal{L}(\mathcal{C}; [P_1(K)]^2)).$\(^1\)

\(^1\) $\mathcal{L}(X, Y)$ denotes the space of linear bounded mappings of $X$ into $Y$. 50
Proof. The numbers \( \alpha_i, \beta_i \) are determined uniquely by the system \((j)\). Indeed, the (Gramm's) matrix \( A^{(i)} = \left[ \begin{array}{c} \lambda^{(i)}_j, \lambda^{(i)}_k \end{array} \right] \), \( j, k = 1, 2 \) is regular. The lemma 2.1 yields the existence and uniqueness of \( w \in [P_1(K)]^2 \) such that
\[
T_i w(a_i) = \alpha_i, \quad T_i w(a_{i+1}) = \beta_i.
\]
Let us set \( \Pi u = w \). The mapping \( \Pi \) is obviously linear. Let \( u_n \to u \) in \( W \) and denote by \( \alpha^n_i, \beta^n_i \) and \( \alpha_i, \beta_i \) the solutions of \((j)\) with the left-hand sides \( [T_i u_n, \lambda^{(i)}_k] \), and \( [T_i u, \lambda^{(i)}_k] \), respectively. Using the trace theorem and the Cramer's rule, we obtain that \( \alpha^n_i \to \alpha_i \), \( \beta^n_i \to \beta_i \). The rest of the proof is obvious as well as the fact that \( \Pi \in L^2(C; [P_1(K)]^2) \).

Q.E.D.

By virtue of Theorem 2.1 there exists a \( c > 0 \) such that
\[
\| \Pi u \|_C \leq c \| u \|_C.
\]

In the following we present an estimate for \( c \).

**Theorem 2.2.** Let \( \Pi \) be defined through \((j), (jj)\). Then
\[
\| \Pi u \|_C \leq \frac{6 \sqrt{2}}{\sin \alpha} \| u \|_C,
\]
holds for all \( u \in C \), where \( \alpha \) is the minimal interior angle of \( K \).

**Proof.** A direct calculation yields that
\[
\det A^{(i)} = \frac{1}{i^2 j^2}, \quad | \alpha_i | \leq 3 \sqrt{2} \| u \|_C, \quad | \beta_i | \leq 3 \sqrt{2} \| u \|_C.
\]
Next from \((jj)\) it follows e.g. for \( \Pi u(a_2) = (w_1(a_2), \lambda^{(i)}_k(a_2)) \)
\[
| w_1(a_2) | = \left| \frac{\beta_1 n^{(1)}_1 - n^{(2)}_1}{\alpha_2 n^{(2)}_2 - n^{(2)}_2} \right| \leq \frac{6 \sqrt{2}}{\sin \alpha} \| u \|_C,
\]
because the \( \det (n^{(1)}, n^{(2)}) \) is equal to the sinus of the angle between \( n^{(1)} \) and \( n^{(2)} \). Similar estimates hold for the remaining values of \( w \) at the vertices. Q.E.D.

Let us define
\[
\mathcal{M}(K) = \{ v = (v_1, v_2), v_j \in P_1(K), j = 1, 2, \div v = 0 \}.
\]
It is easy to see that \( \dim \mathcal{M}(K) = 5 \),
\[
\mathcal{M}(K) = \{ \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ x \\ y \\ x \\ -y \end{array} \right) \},
\]
(2.6) \( v \in \mathcal{M}(K) \iff v = (\gamma_1 + \gamma_2 x + \gamma_3 y, \delta_1 + \delta_2 x - \gamma_2 y) \).
Lemma 2.2.

\( \mathbf{v} \in \mathcal{M}(K) \iff \mathbf{v} \in [P_1(K)]^2 \& \int_{\partial K} \mathbf{v} \cdot \mathbf{n} \, d\Gamma = 0. \)

Proof. Let \( \mathbf{v} \in \mathcal{M}(K) \). Then

\[ \text{div} \mathbf{v} = 0 \quad \text{on} \ K \Rightarrow \int_{\partial K} \mathbf{v} \cdot \mathbf{n} \, d\Gamma = 0 \]

by virtue of the Green's theorem.

Let \( \mathbf{v} = (\gamma_1 + \gamma_2 x + \gamma_3 y, \delta_1 + \delta_2 x + \delta_3 y) \) and

\( (2.7) \int_{\partial K} \mathbf{v} \cdot \mathbf{n} \, d\Gamma = \int_K \text{div} \mathbf{v} \, dx \, dy = 0. \)

Then \( \text{div} \mathbf{v} = \gamma_2 + \delta_3 \) and (2.7) yields

\( (\gamma_2 + \delta_3) \, \text{mes} \, K = 0 \Rightarrow \gamma_2 = -\delta_3. \)

Using also (2.6) we obtain that \( \mathbf{v} \in \mathcal{M}(K) \). Q.E.D.

Lemma 2.3.

\( \mathbf{v} \in \mathcal{M}(K) \iff \mathbf{v} \in [P_1(K)]^2 \& \sum_{i=1}^3 (\alpha_i + \beta_i) \, l_i = 0, \)

where \( \alpha_i = T_i \mathbf{v}(a_i), \beta_i = T_i \mathbf{v}(a_{i+1}). \)

Proof. We may write

\[ T_i \mathbf{v} = T_i \mathbf{v}(a_i) \lambda_1^{(i)} + T_i \mathbf{v}(a_{i+1}) \lambda_2^{(i)} = \alpha_i \lambda_1^{(i)} + \beta_i \lambda_2^{(i)}, \]

\[ \int_{\partial K} \mathbf{v} \cdot \mathbf{n} \, d\Gamma = \sum_{i=1}^3 \int_{a_i a_{i+1}} T_i \mathbf{v} \, d\Gamma = \sum_{i=1}^3 (\alpha_i + \beta_i) \, l_i \]

Consequently,

\[ \int_{\partial K} \mathbf{v} \cdot \mathbf{n} \, d\Gamma = 0 \iff \sum_{i=1}^3 (\alpha_i + \beta_i) \, l_i = 0 \]

and from Lemma 2.2 the assertion follows. Q.E.D.

Let us set

\( (2.8) \quad \mathbf{U} = \{ \mathbf{v} \in \mathbf{W}, \ \text{div} \mathbf{v} = 0 \} \)

Theorem 2.3. Let the mapping \( \Pi \) be defined through the relations (j), (jj), as in Theorem 2.1. Then \( \Pi \in \mathcal{L}(\mathbf{U}; \mathcal{M}(K)) \) and

\( (2.9) \quad \Pi \mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in [P_1(K)]^2. \)
Proof. Adding the equations (j) for \( k = 1, 2 \) yields, by virtue of (2.2):

\[
\int_{a_i, a_{i+1}} T_i u \, ds = [T_i u, \lambda_1^{(i)} + \lambda_2^{(i)}], = \alpha [\lambda_1^{(i)}, 1], + \beta [\lambda_2^{(i)}, 1], = \frac{1}{2} l_i (\alpha_i + \beta_i),
\]

\[
u \in U \Rightarrow 0 = \int_{ \partial K } u \cdot n \, d\Gamma = \sum_{i=1}^{3} \int_{a_i, a_{i+1}} T_i u \, d\Gamma = \frac{3}{2} \sum_{i=1}^{3} (\alpha_i + \beta_i) l_i.
\]

Hence \( \nu \in \mathcal{M}(K) \) follows from Lemma 2.3. The mapping \( \Pi \) is linear and continuous, being the restriction of \( \mathbf{W} \) onto \( U \).

Let \( \nu \in [P_h(K)]^2 \). A direct calculation from (j) implies \( \alpha_i = T_i \nu(a_i), \beta_i = T_i \nu(a_{i+1}) \).

Then using (jj) and Lemma 2.1 we obtain \( T_i \nu = \nu \). Q.E.D.

**Theorem 2.4.** Let \( u \in C^2(K) \). Then

\[
(2.10) \quad \| u - \Pi u \|_C \leq 4 \left( 1 + \frac{6 \sqrt{2}}{\sin \alpha} \right) h^2 \| u \|_{C^2(K)},
\]

where \( h = \text{diam } K \) and \( \alpha \) is the minimal interior angle of \( K \).

Proof. Let \( x_0 \in K \) be an arbitrary point. Taylor’s theorem implies for \( x \in K \)

\[
(2.11) \quad u(x) = u(x_0) + D u(x_0) (x - x_0) + D^2 u(\Theta) (x - x_0)^2,
\]

where \( \Theta \in x_0 x \).

Applying \( \Pi \) to (2.11), using its linearity and (2.9), we obtain

\[
\Pi u(x) = \Pi u(x_0) + \Pi(D u(x_0)) (x - x_0) + \Pi(D^2 u(\Theta)) (x - x_0)^2 = u(x_0) + D u(x_0) (x - x_0) + \Pi(D^2 u(\Theta)) (x - x_0)^2.
\]

Consequently, we have by virtue of (2.5)

\[
\| u - \Pi u \|_C \leq \| D^2 u(\Theta) (x - x_0)^2 \|_C + \Pi D^2 u(\Theta) (x - x_0)^2 \|_C \leq 4 \left( 1 + \frac{6 \sqrt{2}}{\sin \alpha} \right) h^2 \| u \|_{C^2(K)}. \quad \text{Q.E.D.}
\]

Let \( \Omega \in R_2 \) be a bounded polygonal domain, \( h \in (0, 1) \), \( \mathcal{F}_h \) a triangulation of \( \Omega \), satisfying the usual requirements concerning the mutual position of two triangles. Suppose that

\[
h = \text{max diam } K \quad \forall K \in \mathcal{F}_h.
\]

Denote by \( \Pi_K \) the mapping, defined on \( K \in \mathcal{F}_h \) by the conditions (j), (jj) of Theorem 2.1. Let \( K, K' \in \mathcal{F}_h \) be two adjacent triangles, with a common side \( a_i a_{i+1} \) and \( \nu \in W^{1,2}(K \cup K') \). The function \( T_i \nu \), defined by (2.1) with respect to the triangle \( K \)

53
(or \( K' \)) will be denoted by \( T_{i,K} v \) (or \( T_{i,K'} v \), respectively). We say that the condition \((R)\) is satisfied on the common side \( a_i a_{i+1} \), if

\[
T_{i,K} v + T_{i,K'} v = 0 \quad \text{on } a_i a_{i+1}.
\]

(2.12)

Let us define

\[
U(\Omega) = \{ v \in [W^{1,2}(\Omega)]^2, \ \text{div} \ v = 0 \},
\]

\( \mathcal{N}_h(\Omega) = \{ v, v|_K \in \mathcal{M}(K) \ \forall K \in \mathcal{T}_h, \ (R) \) is satisfied on each common side of any pair \( K, K' \) of adjacent triangles of \( \mathcal{T}_h \).\)

For \( v \in U(\Omega) \) we define a mapping \( r_h \) by the relations

\[
(2.13) \quad r_h v|_K = \Pi_K v \quad \forall K \in \mathcal{T}_h.
\]

We say that a family \( \{ \mathcal{T}_h \} \), \( h \in (0, 1) \) of triangulations of \( \Omega \) is regular, if there exists a constant \( \alpha_0 > 0 \) independent of \( h \) and such that all interior angles of the triangles of \( \mathcal{T}_h \in \{ \mathcal{T}_h \} \) are not less than \( \alpha_0 \).

**Theorem 2.5.** Let \( \{ \mathcal{T}_h \} \), \( h \in (0, 1) \) be a regular family of triangulations of \( \Omega \). Then

\[
(2.14) \quad r_h \in L^2(U(\Omega); \mathcal{N}_h(\Omega)),
\]

\[
(2.15) \quad \| u - r_h u \|_{[L_2(\Omega)]^2} \leq c h^2 \| u \|_{[C^2(\Omega)]^2} \quad \forall u \in [C^2(\Omega)]^2,
\]

where \( c \) does not depend on \( h, u \).

**Proof.** With regard to Theorems 2.1, 2.3, (2.14) will follow, if the condition \((R)\) (2.12) is verified. Let \( K, K' \in \mathcal{T}_h \), \( K \cap K' = a_i a_{i+1} \). As \( T_{i,K}(\Pi_K u), T_{i,K'}(\Pi_K u) \in P_1(a_i a_{i+1}) \), it suffices to show that

\[
T_{i,K}(\Pi_K u)(a_j) + T_{i,K'}(\Pi_K u)(a_j) = 0 \quad \text{for } j = i, i + 1.
\]

The latter equation, however, is an immediate consequence of \((j), (jj)\), because \( n^{(i)}_K = -n^{(i)}_{K'} \), \( T_{i,K} u = -T_{i,K'} u \) and

\[
\Pi_K u(a_i) \cdot n^{(i)}_K = \alpha_i, \quad \Pi_K u(a_i) \cdot n^{(i)}_{K'} = -\alpha_i,
\]

\[
\Pi_K u(a_{i+1}) \cdot n^{(i)}_K = \beta_i, \quad \Pi_K u(a_{i+1}) \cdot n^{(i)}_{K'} = -\beta_i.
\]

As the estimate (2.15) is concerned, we obtain, making use of Theorem 2.4

\[
\| u - r_h u \|_{[L_2(\Omega)]^2} \leq \sum_{K \in \mathcal{T}_h} \| u - r_h u \|_{[L_2(K)]^2}^2 = \sum_{K \in \mathcal{T}_h} \| u - \Pi_K u \|_{[L_2(K)]}^2 \leq \sum_{K \in \mathcal{T}_h} \text{mes} K \cdot C^2 h^4 \cdot \| u \|_{[C^2(K)]}^2 \leq C^2 h^4 \cdot \| u \|_{[C^2(\Omega)]}^2,
\]

where \( C = 4 \left(1 + \frac{\sqrt{2}}{\sin \alpha_0} \right) \). Q.E.D.
Remark 2.1 Any function \( v \in \mathcal{N}_h(\Omega) \) satisfies the equation \( \text{div} \ v = 0 \) on \( \Omega \) in the sense of distributions. Indeed, let us take \( \varphi \in \mathcal{D}(\Omega) \) (i.e., infinitely differentiable function with compact support in \( \Omega \)). Then

\[
\langle \text{div} \ v, \varphi \rangle = -\int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, dx \, dy = -\sum_{K \in \mathcal{F}_h} \int_{K} \mathbf{v} \cdot \nabla \varphi \, dx \, dy = 0,
\]

if we use integration by parts and the definition of \( \mathcal{N}_h(\Omega) \) (condition (R)).

3. APPLICATION OF SUBSPACES \( \mathcal{N}_h(\Omega) \) TO THE DUAL VARIATIONAL FORMULATION

Consider a polygonal domain \( \Omega \subset \mathbb{R}^2 \). Let \( \{ \mathcal{F}_h \}, \ h \in (0, 1), \) be a regular family of triangulations of \( \Omega \), satisfying moreover the following requirement: if a part of \( \Gamma_g \) belongs to a side of \( K \in \mathcal{F}_h \), then \( \Gamma_g \) covers the whole side.

Define

\[
V_h = \mathcal{N}_h(\Omega) \bigcap H_2 = \{ \mathbf{v} \in \mathcal{N}_h(\Omega), \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_g \}.
\]

Let \( \chi^0 \in H_2 \) be such that

\[
\Phi(\chi^0) = \min_{\chi \in H_2} \Phi(\chi)
\]

and \( \chi^0_h \in V_h \) such that

\[
\Phi(\chi^0_h) = \min_{\chi \in V_h} \Phi(\chi).
\]

Theorem 3.1.\(^1\) Let \( \chi^0 \in [C^2(\overline{\Omega})]^2 \). Then for any regular family of triangulations \( \{ \mathcal{F}_h \} \) it holds

\[
\| \chi^0 - \chi_h^0 \|_{H} \leq c h^2 \| \chi^0 \|_{C^2(\overline{\Omega})}^2,
\]

where \( c \) is independent of \( h, \chi^0 \).

Proof. We shall verify that \( r_h \chi^0 \in V_h \).

\[
\chi^0 \in [C^2(\overline{\Omega})]^2 \bigcap H_2 \iff \int_{\Omega} \sum_{i=1}^{2} \frac{\partial \mathbf{v}}{\partial x_i} \, dx = 0 \quad \forall \mathbf{v} \in V \iff
\]

\[
\iff \text{div} \ \chi^0 = 0 \text{ in } \Omega, \ \chi^0 \cdot \mathbf{n} = 0 \text{ on } \Gamma_g.
\]

Hence \( \chi^0 \in \mathcal{U}(\Omega) \) and Theorem 2.5 yields that \( r_h \chi^0 \in \mathcal{N}_h(\Omega) \). Consequently, it suffices to prove \( (r_h \chi^0) \cdot \mathbf{n} = 0 \) on \( \Gamma_g \). Let \( a_i a_{i+1} \subset \Gamma_g \) be a side of a boundary triangle

\(^1\) Using the Bramble-Hilbert Lemma, one can prove that \( \| \chi^0 - \chi_h^0 \|_{H} \leq c h^2 \| \chi^0 \|_{L_2, \Omega} \) if \( \chi^0 \in [W^{2,2}(\Omega)]^2 \), where \( \| \cdot \|_{L_2, \Omega} \) denotes the seminorm of second derivatives.
Let \( K \in \mathcal{T}_h \). As \( \chi^0 \cdot n^{(i)} = 0 \) on \( a_i a_{i+1} \), from (j) \( \chi_i = \beta_i = 0 \) follows and (jj) results in
\[
\Pi_K \chi^0(a_i) \cdot n^{(i)} = \Pi_K \chi^0(a_{i+1}) \cdot n^{(i)} = 0.
\]
Consequently, \((\Pi_K \chi^0) \cdot n^{(i)} = 0\) on \( a_i a_{i+1} \) and \( r_h^0 \in V_h \). Using Theorem 1.2 and 2.5, we obtain
\[
\| \chi^0 - \chi^0_h \|_H \leq c_1 \| \chi^0 - r_h^0 \|_H \leq ch^2 \| \chi^0 \|_{(C^2(\Omega))^2}.
\] Q.E.D.

**Corollary.** Let the suppositions of Theorem 3.1 hold. Then for \( \chi^0 = \tilde{\chi} + \chi^0 \), \( \lambda^0_h = \tilde{\chi} + \chi^0_h \) we have
\[
\| \lambda^0 - \lambda^0_h \|_H = O(h^2).
\]

**Proof.** Obviously, we have
\[
\| \lambda^0 - \lambda^0_h \|_H = \| \chi^0 - \chi^0_h \|_H
\]
and we make use of Theorem 3.1. Q.E.D.

Let us recall once more the transformation of the problem \( \mathcal{S} (\lambda) = \min \) on the set \( A_{f,g} \) into the problem (3.1). We supposed that an element \( \tilde{\chi} \in A_{f,g} \) is available. In praxis, however, difficulties may occur in finding this function. Therefore, let us suppose that we are able to construct a \( \tilde{\chi} \in A_{f,g} \), where \( g \) is close to \( g \) in some sense. Let us solve the problem
\[
(3.2) \quad \Phi_\gamma (\chi^0) = \min_{\chi \in H_2} \Phi_\gamma (\chi)
\]
where
\[
\Phi_\gamma (\chi) = \frac{1}{2} \| \chi \|_H^2 + \int_\Omega \sum_{i,j=1}^n b_{ij} \chi_i \chi_j \, dx - B(\chi, \bar{u}).
\]
Let us set
\[
(3.3) \quad \lambda^0 = \tilde{\chi} + \chi^0
\]
(3.4) \quad \lambda^0_\gamma = \tilde{\chi} + \chi^0_\gamma
and seek an estimate for \( \| \lambda^0 - \lambda^0_\gamma \|_H \).

**Theorem 3.2.** Let \( g, \gamma \in L_2(\Gamma_g) \). Then
\[
\| \lambda^0 - \lambda^0_\gamma \|_H \leq c \| g - \gamma \|_{L_2(\Gamma_g)}.
\]

**Proof.** The principle of minimum complementary energy (Theorem 1.1) implies that \( \lambda^0 = \lambda(u) \), where \( \lambda(u) \) is defined by (1.8) and \( u \) is the weak solution of the primal problem (1.3). Similarly, \( \lambda^0_\gamma = \lambda(u_\gamma) \), where \( u_\gamma \) is the weak solution of the modified problem (1.3) with \( g \) replaced by \( \gamma \) on \( \Gamma_g \). From (1.4) and (1.5) it follows
\[
\| u - u_\gamma \|_{W^{1,2}(\Omega)} \leq c \| g - \gamma \|_{L_2(\Gamma_g)}.
\]
56
Finally, we have
\[ \| \lambda^0 - \lambda^0 \|_H = \| \lambda(u - u_\gamma) \|_H \leq c \| u - u_\gamma \|_{W^{1,2}(\Omega)} \leq c' \| g - \gamma \|_{L^2(r_\alpha)}. \] Q.E.D.

Let \( \Gamma_\gamma = \bigcup_{j=1}^m a_j a_{j+1}. \) Suppose that we have found \( \lambda^{(1)} \in H \) such that \( \text{div} \, \lambda^{(1)} = -f, \) (note that this problem is easy to solve by integration) and set
\[ \tilde{g} = g - \lambda^{(1)} \cdot n \text{ on } \Gamma_\gamma. \]

Let \( \tilde{g} \) be such that the values \( \tilde{g}(a_j) \) are well defined for \( j = 1, 2, \ldots, m + 1 \) and find a function \( \tilde{\lambda}^{(2)} = \mathcal{A}_h(\Omega) \) such that at all vertices \( a_j \in \Gamma_\gamma \)
\[ \varphi(a_j) \equiv (\tilde{\lambda}^{(2)} \cdot n)(a_j) = \tilde{g}(a_j), \quad j = 1, 2, 3, \ldots, m + 1. \]

In other words, the function \( \varphi(x) = (\tilde{\lambda}^{(2)} \cdot n)(x) \) represents the Lagrange linear interpolate of the function \( \tilde{g} \) on every \( a_j a_{j+1} \in \Gamma_\gamma. \)

If we set \( \tilde{\lambda} = \lambda^{(1)} + \tilde{\lambda}^{(2)}, \) then
\[ \tilde{\lambda} \in \Lambda_{f, \gamma}, \quad \gamma = \varphi + \lambda^{(1)} \cdot n, \]
because
\[ B(\lambda^{(1)} + \tilde{\lambda}^{(2)}, v) = \int_{\Omega} fv \, dx + \int_{r_\alpha} \gamma v \, d\Gamma \quad \forall v \in V \]
follows from the definition of \( \lambda^{(1)}, \tilde{\lambda}^{(2)}, \) integrating by parts.

Remark 3.1. The problem to find \( \tilde{\lambda}^{(2)}, \) satisfying the conditions mentioned above, is a problem of linear algebra. If \( \alpha_j, \beta_j \) have the sense of Lemma 2.3, i.e. \( \alpha_j = T_j \lambda^{(2)}(a_j), \beta_j = T_j \tilde{\lambda}^{(2)}(a_j) \) on the side \( a_j a_{j+1} \) of a triangle \( K \in \mathcal{T}_h, \) then we have to solve the system:
\[ \sum_{j=1}^m (\alpha_j + \beta_j) l_j = 0 \quad \forall K \in \mathcal{T}_h. \]

Moreover, to satisfy also the condition (R), we choose the values \( \alpha_j, \beta_j \) on the common side of any two adjacent triangles such that (2.12) holds, i.e., the corresponding parameters differ by the sign only. Finally, at the vertices on \( \Gamma_\gamma, \) the values \( \alpha_j, \beta_j \) must equal to the values of \( \tilde{g}. \)

Let us derive an estimate for \( \| \gamma - \varphi \|_{L^2(r_\alpha)}. \) We have
\[
\| \gamma - \varphi \|_{L^2(r_\alpha)}^2 = \sum_{j=1}^m \| \varphi + \lambda^{(1)} \cdot n - \varphi \|_{L^2(a_j a_{j+1})}^2 = \\
= \sum_{j=1}^m \| \tilde{\lambda} g - \tilde{\lambda}(\lambda^{(1)} \cdot n) + \lambda^{(1)} \cdot n - g \|_{L^2(a_j a_{j+1})}^2 \leq \\
\leq 2 \sum_{j=1}^m \left( \| \tilde{\lambda} g - \varphi \|_{L^2(a_j a_{j+1})}^2 + \| \lambda^{(1)} \cdot n - \tilde{\lambda}(\lambda^{(1)} \cdot n) \|_{L^2(a_j a_{j+1})}^2 \right),
\]
where \( \tilde{\lambda} v \) denotes the linear Lagrange interpolate of \( v \) on the segment \( a_j a_{j+1}. \)
If e.g. \( g, \lambda^{(1)} \cdot n \in W^{2,2}(\Gamma_m) \), where \( \Gamma_m \) is any side of the polygon \( \Gamma_g \), then Theorem 3.2 yields
\[
\| \lambda^0 - \lambda_\gamma^0 \|_H \leq c \| g - \gamma \|_{L^2(\Gamma_g)} \leq c h^2 \left( \sum \| g \|_{W^{2,2}(\Gamma_m)} + \sum \| \lambda^{(1)} \cdot n \|_{W^{2,2}(\Gamma_m)} \right).
\]

Let \( \chi_{\gamma,h}^0 \in V_h \) be such that
\[
\Phi_s(\chi_{\gamma,h}^0) = \min_{\chi \in V_h} \Phi_s(\chi)
\]
and let us set
\[
\lambda_{\gamma,h}^0 = \overline{\lambda} + \chi_{\gamma,h}^0.
\]

Then we have the following error estimate

**Theorem 3.3.** If \( g, \lambda^{(1)} \cdot n \in W^{2,2}(\Gamma_m) \) for any side \( \Gamma_m \) of \( \Gamma_g \) and \( \chi_{\gamma}^0 \in [C^2(\Omega)]^2 \),
then for any regular family of triangulations
\[
\| \chi^0 - \chi_{\gamma,h}^0 \|_H \leq C h^2,
\]
where the constant \( C \) is independent of \( h \).

**Proof.** We may write
\[
\| \chi^0 - \chi_{\gamma,h}^0 \|_H \leq \| \chi^0 - \chi_{\gamma}^0 \|_H + \| \chi_{\gamma}^0 - \chi_{\gamma,h}^0 \|_H,
\]
where \( \chi_{\gamma}^0 \) was introduced by (3.4). Using (3.5) and Corollary to Theorem 3.1 (applied to the modified problems (3.2), (3.6)), we obtain the bounds for both terms of the right-hand side. Q.E.D.

4. A POSTERIORI ERROR ESTIMATES, THE HYPERCIRCLE METHOD

Suppose that, besides \( \chi_{\gamma}^0 = \overline{\lambda} + \chi_{\gamma}^0 \), we have also found a Ritz-Galerkin approximate solution \( u_{h,\ast}^0 \) of the primal problem (1.3), i.e. \( u_{h,\ast}^0 \in V_{h,\ast} \), such that
\[
\mathcal{L}(u_{h,\ast}^0) = \min_{u \in V_{h,\ast}} \mathcal{L}(u)
\]
and \( V_{h,\ast} \subset V \) is a finite-dimensional subspace of finite elements.

Then we obtain the following a posteriori estimates of errors.

**Lemma 4.1.** Let \( u \) and \( u_{h,\ast}^0 \) be the solutions of the problem (1.3) and (4.1), respectively. Let \( \lambda_h^0 = \overline{\lambda} + \chi_h^0 \), where \( \overline{\lambda} \in A_{f,g} \) and \( \chi_h^0 \) is the solution of the problem (1.12). Define \( \lambda(v) \) for any \( v \in W^{1,2}(\Omega) \) by the relation (1.9). Then it holds
\[
\| u_{h,\ast}^0 - u \|_{W^{1,2}(\Omega)} \leq C \| \lambda(u_{h,\ast}^0) - \lambda_h^0 \|
\]
and
\[
\| \lambda_h^0 - \lambda(u) \| \leq C \| \lambda(u_{h,\ast}^0) - \lambda_h^0 \|.
\]

\(^1\) It is also sufficient if \( \chi_{\gamma}^0 \in [W^{2,2}(\Omega)]^2 \) only.
Proof. As \( \lambda(u^0_h - u) = \lambda(u^0_h) - \lambda(u) \in H_1 \) and \( \lambda(u) - \lambda^0_h \in H_2 \), by virtue of Lemma 1.1 we may write

\[
(4.4) \quad \| \lambda(u^0_h) - \lambda^0_h \|_H^2 = \| \lambda(u^0_h - u) + \lambda(u) - \lambda^0_h \|_H^2 = \| \lambda(u^0_h - u) \|_H^2 + \| \lambda(u) - \lambda^0_h \|_H^2.
\]

Hence (4.3) follows from (4.4) and (1.6'). An analogue of (1.8) yields that

\[
\| \lambda(u^0 - u) \|_H^2 \geq c \| u^0_h - u \|_{W^{1,2}(\Omega)}^2
\]

and (4.2) follows, making use of (4.4) and (1.6'). Q.E.D.

Lemma 4.2. (Hypercircle). Let the assumptions of Lemma 4.1 hold. Then

\[
(4.5) \quad \frac{1}{2}(\lambda(u^0_h) + \lambda^0_h) - \lambda(u) = \frac{1}{2}\| \lambda(u^0_h) - \lambda^0_h \|_H
\]

Proof. We may write, using Lemma 1.1 and (4.4), that

\[
\| \lambda(u^0_h) + \lambda^0_h - 2\lambda(u) \|_H^2 = \| \lambda(u^0_h - u) + \lambda^0_h - \lambda(u) \|_H^2 = \| \lambda(u^0_h - u) \|_H^2 + \| \lambda^0_h - \lambda(u) \|_H^2 = \| \lambda(u^0_h) - \lambda^0_h \|_H^2.
\]

Q.E.D.

Remark 4.1. The equation (4.5) has the following geometrical sense: the solution \( \lambda(u) \in H \) lies on a hypercircle, the center of which is at the point \( \frac{1}{2}(\lambda(u^0_h) + \lambda^0_h) \) and the diameter equals the distance \( \| \lambda(u^0_h) - \lambda^0_h \|_H \).

Note that if the center of the hypercircle is taken for an approximation of \( \lambda(u) \), then the error measured in \( H \) is equal to the radius of the hypercircle, whereas the errors of both \( \lambda^0_h \) and \( \lambda(u^0_h) \) can only be estimated from above by the diameter, as an immediate consequence of (4.4).

APPENDIX

Let us consider the following mixed boundary value problem, including the Newton condition:

\[
(\text{A1}) \quad \mathcal{A} u = f \quad \text{in} \quad \Omega
\]

\[
\begin{array}{c}
u = \bar{u} \quad \text{on} \quad \Gamma_u \\
\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} n_i = g \quad \text{on} \quad \Gamma_g, \\
\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} n_i + au = \gamma \quad \text{on} \quad \Gamma_N,
\end{array}
\]

59
where the boundary $\Gamma$ consists of at most four disjoint parts

$$
\Gamma = \Gamma_u \cup \Gamma_g \cup \Gamma_N \cup \mathcal{R},
$$

$\Gamma_N \neq \emptyset$ open in $\Gamma$, $\text{mes}_{n-1} \mathcal{R} = 0$, $\Gamma_u$ and $\Gamma_g$ are either empty or open in $\Gamma$,

$$
f \in L_2(\Omega), \quad g \in L_2(\Gamma_g), \quad \gamma \in L_2(\Gamma_N), \quad \bar{u} \in W^{1,2}(\Omega).
$$

$a_{ij}$ satisfy (1.1), (1.2) and

$$
a \in L_\infty(\Gamma_N), \quad \alpha(x) \geq \alpha_0 > 0
$$

almost everywhere on $\Gamma_N$.

Define again

$$
V = \{v|v \in W^{1,2}(\Omega), \ v = 0 \text{ on } \Gamma_u\}
$$

$$
a(u,v) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_{\Gamma_N} \alpha v \, d\Gamma.
$$

The weak solution of the problem (A 1) is a function $u \in W^{1,2}(\Omega)$, satisfying

$$
u - \bar{u} \in V,
$$

$$
a(u,v) = \int_\Omega f v \, dx + \int_{\Gamma_g} g v \, d\Gamma + \int_{\Gamma_N} \gamma v \, d\Gamma \ \forall v \in V.
$$

It is well known, that a unique weak solution of the problem (A.1) exists and that it minimizes the functional

$$
\mathcal{L}(v) = \frac{1}{2} a(v,v) - \int_\Omega f v \, dx - \int_{\Gamma_g} g v \, d\Gamma - \int_{\Gamma_N} \gamma v \, d\Gamma
$$
on the set $\bar{u} \oplus V$.

**Definition A1.** Let $v \in W^{1/2,2}(\Gamma)$, $v = O$ on $\Gamma_u$. Denote $Zv = w \in V$ an arbitrary extension of $v$ into $\Omega$ (see e.g. [10], p. 103 for the extension). Let $M_{f,g} = [L_2(\Omega)]^n$ be the set of vector-functions $\lambda$, such that the operator $\mathcal{G}_{f,g}(\lambda)$, defined through the relation

$$
\langle \mathcal{G}_{f,g}(\lambda), v \rangle = \int_\Omega (\lambda \text{ grad } w - f w) \, dx - \int_{\Gamma_g} g v \, d\Gamma,
$$

maps $M_{f,g}$ into $W^{-1/2,2}(\Gamma_N)$.  

**Remark 1.** From Definition A1 it follows that

$$
|\langle \mathcal{G}_{f,g}(\lambda), v \rangle| \leq c(\lambda) \|v\|_{W^{1/2,2}(\Gamma_N)}
$$

and the values of $\mathcal{G}_{f,g}(\lambda)$ do not depend on the extension $Z$. 

1) $W^{-1/2,2}(\Gamma_N)$ denotes the space of linear continuous functionals on $W^{1/2,2}(\Gamma_N)$.
Remark 2. Consider \( \varphi \in W^{1/2,2}(\Gamma) \) such that \( \varphi = 0 \) on \( \Gamma_u \cup \Gamma_N \), and denote \( Z_\varphi = \psi \). According to Definition A1 we have

\[
\langle \mathcal{G}_{f,\varphi}(\lambda), \varphi \rangle = \int_\Omega (\lambda \cdot \text{grad} \, \varphi - f \, \varphi) \, dx - \int_{\Gamma_g} \varphi \, d\Gamma = 0
\]

(A2)

Consequently, \( \lambda \in M_{f,\varphi} \) satisfies the following conditions

\[
\text{div} \, \lambda + f = 0 \quad \text{in} \quad \Omega
\]

\[
\sum_{j=1}^n \lambda_j n_j = g \quad \text{on} \quad \Gamma_g,
\]

in a weak sense. In fact, integrating by parts in (A2) formally, we obtain

\[
\langle \mathcal{G}_{f,\varphi}(\lambda), \varphi \rangle = -\int_\Omega (\text{div} \, \lambda + f) \, \varphi \, dx + \int_{\Gamma_g} \left( \sum_{j=1}^n \lambda_j n_j - g \right) \varphi \, d\Gamma = 0.
\]

Definition A2. Let \( A_{f,\varphi} \subseteq M_{f,\varphi} \) be the set of all \( \lambda \in M_{f,\varphi} \) such that

\[
\mathcal{G}_{f,\varphi}(\lambda) \in L_2(\Gamma_N).
\]

\( A_{f,\varphi} \) will be called the set of admissible functions.

Next let \( f \) and \( g \) vary all over the space \( L_2(\Omega) \) and \( L_2(\Gamma_g) \), respectively. Denote

\[
M = \bigcup_{f \in L_2(\Omega), \atop g \in L_2(\Gamma_g)} A_{f,\varphi}.
\]

Then to every \( \lambda \in M \) there exists a pair \( \{f(\lambda), g(\lambda)\} \) \( (f = -\text{div} \, \lambda, g = \lambda_j n_j \text{ on } \Gamma_g \) in a weak sense) such that

\[
G(\lambda) \equiv \mathcal{G}_{f(\lambda),g(\lambda)}(\lambda) \in L_2(\Gamma_N).
\]

It is readily seen that \( M \) is a linear manifold and \( G \) is linear on \( M \).

Let us define a bilinear form

\[
(\lambda, \mu)_M = \int_\Omega \sum_{i,j=1}^n b_{ij} \lambda_i \mu_j \, dx + \int_{\Gamma_N} a^{-1} G(\lambda) G(\mu) \, d\Gamma
\]

on \( M \times M \), where \( b_{ij} \) are the entries of the matrix \( [a^{-1}] \) inverse to \( [a] \). Introducing the norm of graph

\[
\| \lambda \|_M^2 = \sum_{i=1}^n \int_\Omega \lambda_i^2 \, dx + \int_{\Gamma_N} [G(\lambda)]^2 \, d\Gamma,
\]
and using the properties of the coefficients $a_{ij}$, we are led to the inequalities

\[(A5) \quad C_1 \| \lambda \|_M^2 \leq (\lambda, \lambda)_H \leq c_2 \| \lambda \|_M^2.\]

The bilinear form is symmetric, continuous and positive definite, hence it represents a scalar product. The manifold $M$ with the scalar product (A4) will be denoted by $H$.

Let us define

$$H_1 = \{ \lambda \mid \lambda \in M, \exists v \in V : \lambda_i = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j} (i = 1, 2, \ldots, n), \quad \lambda v + G(\lambda) = 0 \text{ on } \Gamma_N \}.$$

$H_2 = \Lambda_{0,0}$.

Then we prove the following

**Lemma A1.**

\begin{itemize}
  \item [a)] $H_1$ and $H_2$ are closed subspaces of $M$.
  \item [b)] $H_1 \perp H_2$.
\end{itemize}

**Proof.** Let $\lambda^m \in H_1$, $\lambda^m \to \lambda$ in $H$. Then we may write (cf. the proof of Lemma 1.1)

$$\| \lambda^m - \lambda^p \|_H^2 \geq \sum_{a} \| \text{grad} (v_m - v_p) \|_H^2 + \alpha_0 \int_{\Gamma_N} (v_m - v_p)^2 d\Gamma \geq c \| v_m - v_p \|_{W^{1,2}(\Omega)}^2$$

using the generalized Friedrichs inequality (see [10]). Hence $v_n \to v$ in $V$, and from (A5) it follows that

$$\lambda = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j}, \quad G(\lambda) = -\lambda v, \quad \lambda \in H_1.$$

Definitions A1 and A2 yield

\[(A6) \quad \langle \mathcal{G}_{0,0}(\lambda^n), v \rangle = \int_{\Omega} G(\lambda^n) v \, d\Gamma = \int_{\Omega} \lambda^n \cdot \text{grad} w \, dx\]

for all $v \in W^{1/2,2}(\Gamma_N)$, $w = Zw \in V$, $\lambda^n \in A_{0,0}$.

If $\lambda^n \to \lambda$ in $H$, by virtue of (A5) $\lambda^n \to \lambda$ in $[L_2(\Omega)]^n$, $G(\lambda^n) \to \lambda$ in $L_2(\Gamma_N)$. Therefore (A6) holds even for $\lambda$, i.e., $\lambda \in A_{0,0} = H_2$.

Let $\lambda \in H_1, \mu \in H_2$. Then we obtain

$$\langle \lambda, \mu \rangle_H = \int_{\Omega} \mu \cdot \text{grad} v \, dx - \int_{\Gamma_N} v \, G(\mu) \, d\Gamma = 0,$$

as $G(\mu) = \mathcal{G}_{0,0}(\mu)$.

**Theorem A1.** (Principle of minimum complementary energy). Let $u$ be the weak solution of (A1). Suppose that $\lambda(\bar{u}) \in M$, and $G(\lambda(\bar{u})) = \gamma - \alpha \bar{u}$.
Then the functional
\[ \mathcal{F}(\lambda) = \frac{1}{2}(\lambda, \lambda)_{H} - (\lambda, \lambda(\bar{u}))_{H} \]
attains its minimum on the set \( A_{f,g} \) of admissible functions, if and only if \( \lambda = \lambda(u) \).
Moreover it holds
\[ (A7) \quad -\mathcal{F}(\lambda(u)) = \mathcal{L}(u) + \int_{\Omega} f\bar{u} \, dx + \int_{\Gamma_{g}} g\bar{u} \, d\Gamma + \frac{1}{2} \int_{\Gamma_{N}} \alpha^{-1} \gamma^{2} \, d\Gamma. \]

Proof. First we show that \( \lambda(u) \in A_{f,g} \). In fact, from the definition of a weak solution it follows
\[ (A8) \quad \int_{\Omega} \lambda(u) \operatorname{grad} v \, dx + \int_{\Gamma_{N}} \alpha v \, d\Gamma = \int_{\Omega} f v \, dx + \int_{\Gamma_{g}} g v \, d\Gamma + \int_{\Gamma_{N}} \gamma v \, d\Gamma \]
\( \forall v \in V \). Consequently, we have
\[ \int_{\Omega} (\lambda(u) \operatorname{grad} v - f v) \, dx - \int_{\Gamma_{g}} g v \, d\Gamma = \int_{\Gamma_{N}} (\gamma - \alpha u) \, v \, d\Gamma = \langle \mathcal{G}_{f,g}(\lambda(u)), v \rangle, \quad \mathcal{G}_{f,g}(\lambda(u)) \equiv G(\lambda(u)) = \gamma - \alpha u \in L_{2}(\Gamma_{N}), \]
hence \( \lambda(u) \in A_{f,g} \).
Denote \( u = \bar{u} + \omega, \, \omega \in V \). Then
\[ (A9) \quad \lambda(\omega) \in H_{1}. \]
Indeed, insert \( \lambda(u) = \lambda(\bar{u}) + \lambda(\omega) \) into (A8), and denote by \( f(\bar{u}) = f(\lambda(\bar{u})) \), \( g(\bar{u}) = g(\lambda(\bar{u})) \) the functions, corresponding with \( \lambda(\bar{u}) \in M \) (see Definition A2). Thus we have
\[ (A10) \quad \int_{\Omega} (\lambda(\omega) \operatorname{grad} v + \lambda(\bar{u}) \operatorname{grad} v) \, dx + \int_{\Gamma_{N}} \alpha v(\bar{u} + \omega) \, d\Gamma = \int_{\Omega} f v \, dx + \int_{\Gamma_{g}} g v \, d\Gamma + \int_{\Gamma_{N}} \gamma v \, d\Gamma. \]
and Definition of \( G(\lambda(\bar{u})) \) yields that
\[ \int_{\Omega} \lambda(\bar{u}) \operatorname{grad} v \, dx = \langle G(\lambda(\bar{u})), v \rangle + \int_{\Omega} f(\bar{u}) v \, dx + \int_{\Gamma_{g}} g(\bar{u}) \, d\Gamma. \]
Substituting into (A10), we obtain
\[ \int_{\Omega} [\lambda(\omega) \operatorname{grad} v - (f - f(\bar{u})) v] \, dx - \int_{\Gamma_{g}} [g - g(\bar{u})] \, v \, d\Gamma + \langle G(\lambda(\bar{u})), v \rangle = \int_{\Gamma_{N}} (\gamma - \alpha \bar{u} + \alpha \omega) \, v \, d\Gamma, \quad \forall v \in V. \]
As $G(\lambda(\bar{u})) = \gamma - \alpha \bar{u} \in L^2(\Gamma_N)$ we deduce

$$G(\lambda(\omega)) = \mathcal{G}_{f,\Phi}(\lambda(\omega)) = -\alpha \omega,$$

where

$$f_1 = f - f(\bar{u}), \quad g_1 = g - g(\bar{u}).$$

Consequently, $\lambda(\omega)$ belongs to $H_1$.

Next let $\lambda \in \Lambda_{f,\Phi}$. Then

$$(A11) \quad \lambda - \lambda(u) \in H_2.$$

In fact, subtracting the relations of Definition A1 for $\lambda$ and $\lambda(u)$, we come to the following equation

$$\int_{\Omega} (\lambda - \lambda(u)) \text{grad} w \, dx = \langle \mathcal{G}_{f,\Phi}(\lambda) - \mathcal{G}_{f,\Phi}(\lambda(u)), v \rangle = \langle \mathcal{G}_{0,0}(\lambda - \lambda(u)), v \rangle.$$

Consequently, $\mathcal{G}_{0,0}(\lambda - \lambda(u)) \in L^2(\Gamma_N)$, $\lambda - \lambda(u) \in \Lambda_{0,0}$.

The rest of the proof is the same as that of Theorem 1.1. being based on (A9), (A11) and the orthogonality ($\beta$) of Lemma A1.

Q.E.D.

Following the line of thought of Section 1, we can introduce an equivalent variational problem (1.11) and establish an analogue of Theorem 1.2. The only change is that the norms $\|\lambda\|$ should be replaced by $\|\lambda\|_M$. Using the subspaces $\mathcal{V}_h(\Omega)$, we set again $V_h = \{\chi \mid \chi \in \mathcal{V}_h(\Omega), \chi \cdot n = 0 \text{ on } \Gamma_g\}$, $h \in (0, 1)$.

Then $V_h \subset H_2$ and an exact analogue of Theorem 3.1 holds, as

$$\chi^0 \in \left[C^2(\overline{\Omega})\right]^2 \cap H_2 \Rightarrow G\chi^0 \equiv \chi^0 \cdot n \in C^2(\mathcal{T}_a; a_{j+1}), \quad \forall a_j a_{j+1} \in \Gamma_N,$$

$$\|\chi^0 \cdot n - (r_h x_0) \cdot n\|_{L^2(\Gamma_N)} \leq c h^2 \|x^0\|_{[C^2(\overline{\Omega})]^2}$$

can be proved easily.

Acknowledgments. The authors are indebted to Doc. J. Nečas, DrSc. for some valuable advices concerning the approach used in this Appendix.

References


64
Je studován „rovnovážný“ model metody konečných prvků při aplikaci po částech (trojúhelnících) lineárních polynomů na řešení kombinované okrajové úlohy v rovině pro eliptickou diferenciální rovnici druhého řádu. Dokazuje se, že je-li řešení dost hladké, přibližná řešení konvergují s rychlosti $h^2$.