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ON KLOTZ'S RESULT ON THE ASYMPTOTIC EFFICIENCY
FOR THE SIGNED RANK TESTS

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1. INTRODUCTION

The most important problem in investigating the asymptotic efficiency in the Bahadur sense of a sequence of statistics $\{S_n\}$ is to find the exponential rate of convergence to zero of the probabilities of large deviations under the hypothesis \mathcal{H} , namely to compute

$$(1.1) \quad \lim_{n \rightarrow \infty} \left[-\frac{1}{n} \log \sup \{Pr(S_n > nr_n | f) : f \in \mathcal{H}\} \right] = K(r), \quad \text{say,}$$

where $\{r_n\}$ is a sequence of constants tending to some $r > 0$, (see, e.g., [1], [3]). If \mathcal{H} and S_n are nonparametric, the limit gets much simpler, and it is

$$(1.2) \quad \lim_{n \rightarrow \infty} \left[-\frac{1}{n} \log P(S_n > nr_n) \right] = K(r),$$

where P indicates the probability measure under \mathcal{H} , and this notation will be kept throughout the paper.

Let X_1, \dots, X_n be a sequence of independent random variables having the same continuous cdf $F(x)$. Let $X_{(1)}, \dots, X_{(n)}$ be the rearrangement of X_1, \dots, X_n ordered by magnitude of their absolute values, i.e. $|X_{(1)}| \leq \dots \leq |X_{(n)}|$, and let R_1^+, \dots, R_n^+ be the corresponding ranks of $|X_1|, \dots, |X_n|$. Let U_1, \dots, U_n denote the signs of $X_{(1)}, \dots, X_{(n)}$. The nonparametric hypothesis \mathcal{H} in the paper consists of all $F(x)$ which are symmetric about 0 but otherwise arbitrary.

Klotz [4] has investigated the limit (1.2) for

$$(1.3) \quad S_n = \sum_{i=1}^n E_{ni} U_i,$$

where E_{ni} , $1 \leq i \leq n$, are the expected values of the i -th smallest order statistics from a sample with cdf $G(x)$ on $(0; \infty)$ satisfying

$$(1.4) \quad \int_0^\infty x^3 dG(x) < \infty .$$

Note that, under \mathcal{H} , the variables U_1, \dots, U_n are independent and

$$(1.5) \quad P(U_i = 1) = P(U_i = -1) = 1/2, \quad 1 \leq i \leq n .$$

The same problem has been explored in [3] for the most general case of the linear signed rank tests

$$S_n = \sum_{i=1}^n \alpha_n(i)/(n+1), \quad R_i^+/(n+1), \quad W_i),$$

from which Klotz's result follows.

In this paper the author will give a different direct proof of one result in [3] which generalized Klotz's one.

2. RESULTS

Theorem. *Let*

$$(2.1) \quad S_n = \sum_{i=1}^n a_{ni} U_i ,$$

with $a_{ni} = a_n(i)$ satisfying

$$(2.2) \quad \int_0^1 |a_n(1 + [nu]) - \varphi(u)| du \rightarrow 0 \quad \text{as } n \rightarrow \infty ,$$

for some $\varphi(u) \in L_1(0; 1)$, where $[\cdot]$ indicates the integer function. Let

$$(2.3) \quad r_n \rightarrow r, \quad 0 < r < \int_0^1 |\varphi(u)| du = M, \quad \text{say} .$$

Then

$$(2.4) \quad \lim_{n \rightarrow \infty} \left[-\frac{1}{n} \log P(S_n > nr_n) \right] = K(r) ,$$

where $K(r)$ is evaluated from

$$(2.5) \quad K(r) = br - \int_0^1 \log \cosh (b \varphi(u)) du$$

with $b > 0$ being a unique solution of

$$(2.6) \quad \int_0^1 \varphi(u) \tanh (b \varphi(u)) du = r .$$

Proof. First note that

$$h(b) = \int_0^1 \varphi(u) \tanh (b \varphi(u)) du = \int_0^1 |\varphi(u)| \tanh (b|\varphi(u)|) du$$

is an increasing function of b , and takes value 0 at $b = 0$ and M at $b = \infty$. Thus for r satisfying (2.3) there is always a unique solution $b > 0$ of (2.6).

Clearly (2.2) implies

$$(2.7) \quad (a) \quad a_n(1 + [nu]) \rightarrow \varphi(u) \text{ in the Lebesgue measure } \mathcal{L} \text{ on } (0, 1)$$

$$(b) \quad \int_0^1 a_n(1 + [nu]) du = \frac{1}{n} \sum_{i=1}^n a_{ni} \rightarrow \int_0^1 \varphi(u) du ,$$

$$(c) \quad \int_0^1 |a_n(1 + [nu])| du = \frac{1}{n} \sum_{i=1}^n |a_{ni}| \rightarrow \int_0^1 |\varphi(u)| du = M , \text{ as } n \rightarrow \infty .$$

By Lebesgue's theorem (cf. [5], Th. 3, p. 137), for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \int_{\Delta} a_n(1 + [nu]) du \right| \leq \int_{\Delta} |a_n(1 + [nu])| du < \varepsilon$$

for all $\Delta \subset (0; 1)$ with $\mathcal{L}(\Delta) < \delta$, and for all $n = 1, 2, \dots$. Hence

$$\frac{1}{n} |a_{ni}| = \int_{(i-1)/n}^{i/n} |a_n(1 + [nu])| du < \varepsilon$$

for $1 \leq i \leq n$ if $n > 1/\delta$. Therefore

$$(2.8) \quad \lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} \left\{ \frac{1}{n} |a_{ni}| \right\} = 0 .$$

Denote

$$(2.9) \quad V_i = a_{ni} U_i , \quad 1 \leq i \leq n .$$

Let $F_i(x)$ be the cdf of V_i under \mathcal{H} . For $b > 0$ and $1 \leq i \leq n$, put

$$p_i = E(e^{bV_i} | \mathcal{H}) = \int_{-\infty}^{\infty} e^{bx} dF_i(x) = \cosh (ba_{ni}) ,$$

and

$$H_i(x) = \frac{1}{p_i} \int_{-\infty}^x e^{by} dF_i(y) .$$

Thus H_i may be considered as a “new” cdf of V_i in accordance with a “new” probability measure Q given by

$$(2.10) \quad \begin{aligned} Q(V_i = a_{ni}) &= e^{ba_{ni}}/2 \cosh(ba_{ni}) = \alpha_i, \quad \text{say,} \\ Q(V_i = -a_{ni}) &= e^{-ba_{ni}}/2 \cosh(ba_{ni}) = 1 - \alpha_i = \beta_i, \quad \text{say.} \end{aligned}$$

Let H be the corresponding “new” cdf of S_n provided V_1, \dots, V_n are independent under Q . Using Feller’s transformation (see (3.9), [2]), one has

$$(2.11) \quad P(S_n \leq x) = \prod_{i=1}^n p_i \int_{-\infty}^x e^{-by} dH(y) = \prod_{i=1}^n \cosh(ba_{ni}) \int_{-\infty}^x e^{-by} dH(y).$$

It follows from (2.10) that

$$(2.12) \quad E(S_n | H) = \sum_{i=1}^n a_{ni}(\alpha_i - \beta_i) = \sum_{i=1}^n a_{ni} \tanh(ba_{ni}) = \mu_n(b) = \mu_n, \quad \text{say,}$$

and

$$(2.13) \quad \text{Var}(S_n | H) = 4 \sum_{i=1}^n a_{ni}^2 \alpha_i \beta_i = \sum_{i=1}^n a_{ni}^2 / \cosh^2(ba_{ni}) = B_n^2(b) = B_n^2, \quad \text{say.}$$

From (2.11) one has

$$(2.14) \quad \begin{aligned} P(S_n > \mu_n - 2B_n) &= \prod_{i=1}^n \cosh(ba_{ni}) \int_{y > \mu_n - 2B_n} e^{-by} dH(y) = \\ &= \prod_{i=1}^n \cosh(ba_{ni}) e^{-b\mu_n} \int_{z > -2B_n} e^{-bz} dH^*(z), \end{aligned}$$

where $H^*(z) = H(z + \mu_n)$ is the “new” cdf of $Z_n = S_n - \mu_n$. Noting that $E(Z_n | H^*) = 0$, $\text{Var}(Z_n | H^*) = B_n^2$, and that e^{-bz} is a decreasing function in z , we get

$$(2.15) \quad \begin{aligned} e^{2bB_n} &\geq \int_{z > -2B_n} e^{-bz} dH^*(z) \geq e^{-2B_n b} \int_{|z| < 2B_n} e^{-bz} dH^*(z) \\ &\geq e^{-2B_n b} (1 - B_n^2/4B_n^2) = (3/4) e^{-2B_n b}, \end{aligned}$$

by Chebyshev’s inequality.

It follows from (2.7), (2.8) and (2.13) that

$$B_n^2/n^2 \leq (1/n^2) \sum_{i=1}^n a_{ni}^2 \leq (1/n^2) \max_{1 \leq i \leq n} |a_{ni}| \sum_{i=1}^n |a_{ni}| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

i.e.

$$(2.16) \quad B_n/n = o(1), \quad \text{as } n \rightarrow \infty.$$

Since $x \tanh (bx)$ and $\log \cosh (bx)$ satisfy Lipschitz's condition, it is easy to see from (2.7) that

$$(2.17) \quad \mu_n/n = (1/n) \sum_{i=1}^n a_{ni} \tanh (ba_{ni}) = \int_0^1 \varphi(u) \tanh (b \varphi(u)) du + o(1),$$

and

$$(2.18) \quad (1/n) \sum_{i=1}^n \log \cosh (ba_{ni}) = \int_0^1 \log \cosh (b \varphi(u)) du + o(1), \quad \text{as } n \rightarrow \infty.$$

Clearly, by (2.3), (2.16), (2.17) and by the note at the beginning of the proof one can choose $b_n > 0$ satisfying

$$(2.19) \quad (1/n) \mu_n(b_n) - (2/n) B_n(b_n) = r_n,$$

or equivalently, as $n \rightarrow \infty$,

$$(2.20) \quad \int_0^1 \varphi(u) \tanh (b_n \varphi(u)) du = r + o(1).$$

Evidently

$$(2.21) \quad b_n \rightarrow b, \quad \text{as } n \rightarrow \infty,$$

where $b < 0$ is a unique solution of (2.6) Finally, the theorem follows from (2.14) to (2.21).

Corollary. Klotz's result mentioned in Section 1 remains true under a weaker assumption on $G(x)$:

$$(2.22) \quad \int_0^\infty x dG(x) < \infty.$$

Proof. Put $\varphi(u) = G^{-1}(u)$ in the Theorem.

References

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Souhrn

O KLOTZOVĚ VÝSLEDKU O ASYMPTOTICKÉ EFICIENCI
ZNAMÉNKOVANÝCH POŘADOVÝCH TESTŮ

NGUYEN-VAN-HO

V článku se odvozuje vzorec pro Bahadurovu eficienci testů symetrie znaménkovanými pořadími. Jde o speciální případ dřívějšího autorova výsledku z [3], zde však je důkaz proveden pomocí odlišné jednodušší metody vhodné pro třídu jednoduchých pořadových statistik. Výsledek článku platí za obecnějších předpokladů než u J. Klotze [4].

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