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THE 0–1 LAW GENERALIZED FOR NON-DENUMERABLE FAMILIES OF EVENTS AND OF σ-ALGEBRAS OF EVENTS

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INTRODUCTION

Let $(\Omega, \mathscr{A}, P)$ be a complete probability space. Let T be an arbitrary set of indices, $T = \{t\}$, such that

$$\text{card } T \geqq \text{card } N\,, \quad \text{where} \quad N = \{1, 2, 3, \ldots\}\,. \tag{1.1}$$

Let $\{A_t, t \in T\} \subset \mathscr{A}$ and $\{\sigma_t, t \in T\}$ be a family of σ-algebras of events in $\mathscr{A}$. Let $\sigma(\cdot)$ denote the σ-algebra generated by $(\cdot)$.

In the case card $T =$ card N, $t = \{t_n\}$, $n \in N$, the following definitions are well-known:

$$\limsup A_{t_n} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_{t_k} \quad (\in \mathscr{A})\,, \tag{1.2}$$

$$\liminf A_{t_n} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{t_k} \quad (\in \mathscr{A})\,, \tag{1.3}$$

$$\limsup \sigma_{t_n} = \bigcap_{n=1}^{\infty} \sigma(\sigma_{t_n}, \sigma_{t_{n+1}}, \sigma_{t_{n+2}}, \ldots) \quad (\text{being a } \sigma\text{-algebra} \subset \mathscr{A})\,. \tag{1.4}$$

It is clear that

$$\liminf A_n = \Omega \setminus \limsup \bar{A}_n\,, \quad \text{where} \quad \bar{A}_n = \Omega \setminus A_n\,. \tag{1.5}$$

The following two theorems are well known (see, e.g. [1], [2], [3], [4]).

The Borel-Cantelli Lemma. If $\{A_n\}$, $n \in N$, is a sequence of independent events in $\mathscr{A}$, then $P(\limsup A_n) = 0$, or $= 1$, according to $\sum_{n=1}^{\infty} P(A_n) < \infty$, or $= \infty$, respectively.

The 0–1 law of Kolmogorov. If $\{\sigma_n\}$, $n \in N$, is a sequence of independent σ-algebras in $\mathscr{A}$, then $\limsup \sigma_n$ is composed of events of probability 0 or 1.

In Section 2 the author will generalize the definitions in (1.2)–(1.4) to the definitions of $\operatorname*{SUP}_T A_t$, $\operatorname*{INF}_T A_t$, and $\operatorname*{SUP}_T \sigma_t$, respectively, for the case (1.1).

In Section 3 there will be given results generalizing the Borel-Cantelli Lemma and the 0 – 1 law of Kolmogorov.

2. GENERAL DEFINITIONS

Let T, N, $\{A_t, t \in T\}$, $\{\sigma_t, t \in T\}$ be given as in Section 1. Let (1.1) be satisfied. Denote

$$S(T) = \{\{t_n\} : n \in N,\ t_n \in T,\ t_i \neq t_j \quad \text{if} \quad i \neq j \in N\}, \tag{2.1}$$

i.e. $S(T)$ is the set of all subsequences $\{t_n\}$ of distinct indices of T.

Let us define:

$$\operatorname*{SUP}_T A_t = \bigcup_{\{t_n\} \in S(T)} \limsup A_{t_n} = \bigcup_{\{t_n\} \in S(T)} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_{t_k}, \tag{2.2}$$

$$\operatorname*{INF}_T A_t = \bigcap_{\{t_n\} \in S(T)} \liminf A_{t_n} = \bigcap_{\{t_n\} \in S(T)} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{t_k}, \tag{2.3}$$

and

$$\operatorname*{SUP}_T \sigma_t = \sigma(\sigma_{\{t_n\}}, \{t_n\} \in S(T)), \tag{2.4}$$

where $\sigma_{\{t_n\}}$ denotes $\limsup \sigma_{t_n}$.

Clearly,

$$\operatorname*{INF}_T A_t = \Omega \setminus \operatorname*{SUP}_T \bar{A}_t. \tag{2.5}$$

The following Lemma shows that the new definitions generalize the ones in (1.2) to (1.4) respectively.

Lemma 1. *If*

$$\operatorname{card} T = \operatorname{card} N, \quad T = \{t_n\}, \quad n \in N, \tag{2.6}$$

then

$$\operatorname*{SUP}_T A_t = \limsup A_{t_n}, \tag{2.7}$$

$$\operatorname*{INF}_T A_t = \liminf A_{t_n}, \tag{2.8}$$

and

$$\operatorname*{SUP}_T \sigma_t = \limsup \sigma_{t_n}. \tag{2.9}$$

Proof. a) Evidently, $\limsup A_{t_n} \subset \underset{T}{\mathrm{SUP}}\, A_t$. Now, let $\omega \in \underset{T}{\mathrm{SUP}}\, A_t$. There exists a subsequence $\{t_{n(k)}\} \in S(T)$ such that $\omega \in \limsup A_{t_{n(k)}}$, by (2.2). On the other hand, $\limsup A_{t_{n(k)}} \subset \limsup A_{t_n}$, by (1.2) and by $\{t_{n(k)}\} \subset \{t_n\}$. Therefore $\underset{T}{\mathrm{SUP}}\, A_t \subset$ $\subset \limsup A_{t_n}$, and (2.7) is proved.

b) (2.8) follows from (1.5), (2.5), and (2.7).

c) Obviously, $\limsup \sigma_{t_n} \subset \underset{T}{\mathrm{SUP}}\, \sigma_t$.

Let $m \in N$ be given. Let $\{t_{n(k)}\} \in S(T)$. Hence $\{t_{n(k)}\} \subset \{t_n\}$ and $n(k) \to \infty$ as $k \to \infty$. Thus there is a $k(m) \in N$ such that $n(k) \geqq m$ for all $k \geqq k(m)$. One has successively

$$\limsup \sigma_{t_{n(k)}} \subset \sigma(\sigma_{t_m}, \sigma_{t_{m+1}}, \sigma_{t_{m+2}}, \ldots)$$

for every $\{t_{n(k)}\} \in S(T)$, by (1.4),

$$\underset{T}{\mathrm{SUP}}\, \sigma_t \subset \sigma(\sigma_{t_m}, \sigma_{t_{m+1}}, \sigma_{t_{m+2}}, \ldots)$$

for every $m \in N$, by (2.4),

$$\underset{T}{\mathrm{SUP}}\, \sigma_t \subset \limsup \sigma_{t_n}\,, \quad \text{by} \quad (1.4)\,.$$

This completes the proof of (2.9).

3. RESULTS

Note that when $\operatorname{card} T \geqq \operatorname{card} N$, $\underset{T}{\mathrm{SUP}}\, \sigma_t$ defined by (2.4) is always a σ-algebra of events in $\mathscr{A}$, while $\underset{T}{\mathrm{SUP}}\, A_t$ or $\underset{T}{\mathrm{INF}}\, A_t$ with $\operatorname{card} T > \operatorname{card} N$ belongs to $\mathscr{A}$ only under some conditions. However it will be proved in Theorem 1 below that one of them is always an event in $\mathscr{A}$ having probability 1 or 0 respectively.

Theorem 1. *Let $(\Omega, \mathscr{A}, P)$ be a complete probability space, and let $\{A_t, t \in T\}$, with T satisfying (1.1), be a family of independent events in $\mathscr{A}$. At least one of the following assertions is always valid*:

(3.1) $$\underset{T}{\mathrm{SUP}}\, A_t \in \mathscr{A}\,, \quad P(\underset{A}{\mathrm{SUP}}\, A_t) = 1\,,$$

(3.2) $$\underset{T}{\mathrm{INF}}\, A_t \in \mathscr{A}\,, \quad P(\underset{T}{\mathrm{INF}}\, A_t) = 0\,.$$

More precisely,

(i) (3.1) *is satisfied if there exists* $\{t_n\} \in S(T)$ *such that*

(3.3) $$\sum_{n=1}^{\infty} P(A_{t_n}) = \infty\,,$$

(ii) (3.2) *is satisfied if there exists* $\{t_n\} \in S(T)$ *such that*

$$\sum_{n=1}^{\infty} P(A_{t_n}) < \infty \quad \text{or} \quad \sum_{n=1}^{\infty} (1 - P(A_{t_n})) = \infty\,, \tag{3.4}$$

(iii) *both* (3.1) *and* (3.2) *are satisfied if we have* (3.3) *for some* $\{t_n\} \in S(T)$ *as well as* (3.4) *for some* $\{t'_n\} \in S(T)$.

Proof. a) If (3.3) is satisfied for some $\{t_n\} \in S(T)$, then from the Borel-Cantelli Lemma we get $P(\limsup A_{t_n}) = 1$, i.e.

$$P(\Omega) \setminus \limsup A_{t_n}) = 0\,.$$

Since $\limsup A_{t_n} \subset \underset{T}{\mathrm{SUP}}\, A_t$, or equivalently $\Omega \setminus \underset{T}{\mathrm{SUP}}\, A_t \subset \Omega \setminus \limsup A_{t_n}$, one has $\Omega \setminus \underset{T}{\mathrm{SUP}}\, A_t \in \mathscr{A}$ and $P(\Omega \setminus \underset{T}{\mathrm{SUP}}\, A_t) = 0$, by the completeness of the probability space.

Therefore (3.1) is valid.

b) If one of the conditions in (3.4) is satisfied for some $\{t_n\} \in S(T)$, we have then $\sum_{n=1}^{\infty} P(\bar{A}_{t_n}) = \infty$. Now (3.2) follows from (2.5) and the proof above for $\{\bar{A}_t, t \in T\}$.

The following Theorem generalizes the $0-1$ law of Kolmogorov.

Theorem 2. *Let* $\{\sigma_t, t \in T\}$ *with* card $T \geqq$ card N *be a family of independent* σ*-algebras contained in* $\mathscr{A}$. *Then*

$$P(A) = 0 \text{ or } = 1 \quad \textit{for all} \quad A \in \underset{T}{\mathrm{SUP}}\, \sigma_t\,. \tag{3.5}$$

Proof. Denote

$$\mathfrak{M} = \{A : A \in \mathscr{A},\ P(A) = 0 \text{ or } = 1\}\,. \tag{3.6}$$

The $0-1$ law of Kolmogorov implies

$$\mathfrak{M} \supset \sigma_{\{t_n\}} \quad \text{for every} \quad \{t_n\} \in S(T)\,. \tag{3.7}$$

It follows from (3.6) that

$$\begin{aligned} &\text{(a)} \quad A, B \in \mathfrak{M} \Rightarrow A \cup B \in \mathfrak{M}\,,\\ &\text{(b)} \quad A \in \mathfrak{M} \Rightarrow \bar{A} \in \mathfrak{M}\,,\\ &\text{(c)} \quad \Omega \in \mathfrak{M}\,. \end{aligned} \tag{3.8}$$

Hence $\mathfrak{M}$ is an algebra containing the family $(\sigma_{\{t_n\}}, \{t_n\} \in S(T))$. Moreover, $\mathfrak{M}$ is a monotone class. In fact, let $\{A_n\} \subset \mathfrak{M}$, $A_n \uparrow$, then

$$P(\lim \uparrow A_n) = \lim_{n\to\infty} P(A_n) = \begin{cases} 1 & \text{if there is } A_k \text{ such that } P(A) = 1\,,\\ 0 & \text{if } P(A_n) = 0 \quad \text{for all} \quad n \in N\,. \end{cases}$$

Hence $\lim \uparrow A_n \in \mathfrak{M}$. Similarly, one has also $\lim \downarrow A_n \in \mathfrak{M}$ for $A_n \downarrow$ in $\mathfrak{M}$. Therefore $\mathfrak{M}$ is a σ-algebra containing

$$\sigma(\sigma_{\{t_n\}}, \{t_n\} \in S(T)) = \operatorname*{SUP}_T \sigma_t .$$

This completes the proof.

References

[1] *J. Neveu:* Bases mathématiques du calcul des probabilités. Paris, 1964.

[2] *W. Feller:* An introduction to probability theory and its applications. New York, 1966.

[3] *A. Rényi:* Probability theory. Budapest, 1970.

[4] *И. И. Гихман, А. В. Скороход:* Теория случайных процессов. Москва, 1971.

Souhrn

ZÁKON 0–1 ZOBECNĚNÝ PRO NESPOČETNÉ SYSTÉMY JEVŮ A JEVOVÝCH σ-ALGEBER

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Pojmy $\limsup A_n$, $\liminf A_n$ pro posloupnosti množin A_n a pojem $\limsup \sigma_n$ pro posloupnosti σ-algeber σ_n jsou v článku zobecněny pro nespočetné systémy množin, resp. σ-algeber. Na základě těchto zobecněných definic se pak dokazuje určitá slabší obdoba Borelova-Cantelliho lemmatu pro nespočetné systémy množin A_t, $t \in T$, a přímé zobecnění Kolmogorovova $0-1$ zákona pro nespočetné systémy σ-algeber σ_t, $t \in T$.

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