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_Aplikace matematiky_, Vol. 21 (1976), No. 4, 296–300

Persistent URL: [http://dml.cz/dmlcz/103649](http://dml.cz/dmlcz/103649)

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THE 0—1 LAW GENERALIZED FOR NON-DENUMERABLE FAMILIES OF EVENTS AND OF $\sigma$-ALGEBRAS OF EVENTS

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(Received January 27, 1976)

INTRODUCTION

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. Let $T$ be an arbitrary set of indices, $T = \{t\}$, such that

\[(1.1) \quad \text{card } T \geq \text{card } N, \quad \text{where } N = \{1, 2, 3, \ldots\}.
\]

Let $\{A_t, t \in T\} \subset \mathcal{A}$ and $\{\sigma_t, t \in T\}$ be a family of $\sigma$-algebras of events in $\mathcal{A}$. Let $\sigma(\cdot)$ denote the $\sigma$-algebra generated by $\{\cdot\}$.

In the case $\text{card } T = \text{card } N, t = \{t_n\}, n \in N$, the following definitions are well-known:

\[(1.2) \quad \limsup A_{t_n} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_{t_k} \quad (\in \mathcal{A}),
\]

\[(1.3) \quad \liminf A_{t_n} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{t_k} \quad (\in \mathcal{A}),
\]

\[(1.4) \quad \limsup \sigma_{t_n} = \bigcap_{n=1}^{\infty} \sigma(\sigma_{t_n}, \sigma_{t_{n+1}}, \sigma_{t_{n+2}}, \ldots) \quad (\text{being a } \sigma\text{-algebra } \subset \mathcal{A}).
\]

It is clear that

\[(1.5) \quad \liminf A_n = \Omega \setminus \limsup \bar{A}_n, \quad \text{where } \bar{A}_n = \Omega \setminus A_n.
\]

The following two theorems are well known (see, e.g. [1], [2], [3], [4]).

The Borel-Cantelli Lemma. If $\{A_n\}, n \in N$, is a sequence of independent events in $\mathcal{A}$, then $P(\limsup A_n) = 0$, or $= 1$, according to $\sum_{n=1}^{\infty} P(A_n) < \infty$, or $= \infty$, respectively.

The 0—1 law of Kolmogorov. If $\{\sigma_n\}, n \in N$, is a sequence of independent $\sigma$-algebras in $\mathcal{A}$, then $\limsup \sigma_n$ is composed of events of probability 0 or 1.
In Section 2 the author will generalize the definitions in (1.2)—(1.4) to the definitions of $\text{SUP}_T A_t$, $\text{INF}_T A_t$, and $\text{SUP}_T \sigma_t$, respectively, for the case (1.1).

In Section 3 there will be given results generalizing the Borel-Cantelli Lemma and the $0-1$ law of Kolmogorov.

2. GENERAL DEFINITIONS

Let $T, N, \{A_t, t \in T\}, \{\sigma_t, t \in T\}$ be given as in Section 1. Let (1.1) be satisfied. Denote
\[
S(T) = \{\{t_n\} : n \in N, t_n \in T, t_i \neq t_j \text{ if } i \neq j \in N\},
\]
i.e. $S(T)$ is the set of all subsequences $\{t_n\}$ of distinct indices of $T$.

Let us define:
\[
\text{SUP}_T A_t = \bigcup_{\{t_n\} \in S(T)} \limsup_{n \to \infty} A_{t_n},
\]
\[
\text{INF}_T A_t = \bigcap_{\{t_n\} \in S(T)} \liminf_{n \to \infty} A_{t_n},
\]
and
\[
\text{SUP}_T \sigma_t = \sigma(\sigma_{\{t_n\}}, \{t_n\} \in S(T)),
\]
where $\sigma_{\{t_n\}}$ denotes $\limsup_{n} \sigma_{t_n}$.

Clearly,
\[
\text{INF}_T A_t = \Omega \setminus \text{SUP}_T \bar{A}_t.
\]

The following Lemma shows that the new definitions generalize the ones in (1.2) to (1.4) respectively.

Lemma 1. If
\[
\text{card } T = \text{card } N, \quad T = \{t_n\}, \quad n \in N,
\]
then
\[
\text{SUP}_T A_t = \limsup_{n} A_{t_n},
\]
\[
\text{INF}_T A_t = \liminf_{n} A_{t_n},
\]
and
\[
\text{SUP}_T \sigma_t = \limsup_{n} \sigma_{t_n}.
\]
Proof. a) Evidently, \( \limsup_{\omega} A_{\omega_{\omega}} \subset \sup_{T} A_{\omega} \). Now, let \( \omega \in \sup_{T} A_{\omega} \). There exists a subsequence \( \{t_{n(k)}\} \in S(T) \) such that \( \omega \in \limsup_{t_{n(k)}} A_{t_{n(k)}} \), by (2.2). On the other hand, \( \limsup_{A_{t_{n(k)}}} A_{t_{n(k)}} \subset \limsup_{T} A_{t_{n(k)}} \) by (1.2) and by \( \{t_{n(k)}\} \subset \{t_{n}\} \). Therefore \( \sup_{T} A_{\omega} \subset \limsup_{T} A_{\omega} \), and (2.7) is proved.

b) (2.8) follows from (1.5), (2.5), and (2.7).

c) Obviously, \( \limsup_{T} \sigma_{t_{n}} \subset \sup_{T} \sigma_{t} \).

Let \( m \in N \) be given. Let \( \{t_{n(k)}\} \in S(T) \). Hence \( \{t_{n(k)}\} \subset \{t_{n}\} \) and \( n(k) \to \infty \) as \( k \to \infty \). Thus there is a \( k(m) \in N \) such that \( n(k) \geq m \) for all \( k \geq k(m) \). One has successively

\[
\limsup_{t_{n(k)}} \sigma_{t_{n(k)}} \subset \sigma(\sigma_{t_{m}}, \sigma_{t_{m+1}}, \sigma_{t_{m+2}}, \ldots)
\]

for every \( \{t_{n(k)}\} \in S(T) \), by (1.4),

\[
\sup_{T} \sigma_{t} \subset \sigma(\sigma_{t_{m}}, \sigma_{t_{m+1}}, \sigma_{t_{m+2}}, \ldots)
\]

for every \( m \in N \), by (2.4),

\[
\sup_{T} \sigma_{t} \subset \limsup_{T} \sigma_{t_{n}}, \quad \text{by (1.4)}.
\]

This completes the proof of (2.9).

3. RESULTS

Note that when \( \text{card } T \geq \text{card } N \), \( \sup_{T} \sigma_{t} \) defined by (2.4) is always a \( \sigma \)-algebra of events in \( \mathcal{A} \), while \( \sup_{T} A_{t} \) or \( \inf_{T} A_{t} \) with \( \text{card } T > \text{card } N \) belongs to \( \mathcal{T} \) only under some conditions. However it will be proved in Theorem 1 below that one of them is always an event in \( \mathcal{T} \) having probability 1 or 0 respectively.

**Theorem 1.** Let \((\Omega, \mathcal{A}, P)\) be a complete probability space, and let \( \{A_{t}, t \in T\} \), with \( T \) satisfying (1.1), be a family of independent events in \( \mathcal{A} \). At least one of the following assertions is always valid:

\[
(3.1) \quad \sup_{T} A_{t} \in \mathcal{A}, \quad P(\sup_{T} A_{t}) = 1,
\]

\[
(3.2) \quad \inf_{T} A_{t} \in \mathcal{A}, \quad P(\inf_{T} A_{t}) = 0.
\]

More precisely,

(i) (3.1) is satisfied if there exists \( \{t_{n}\} \in S(T) \) such that

\[
(3.3) \quad \sum_{n=1}^{\infty} P(A_{t_{n}}) = \infty,
\]

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(ii) (3.2) is satisfied if there exists \( \{t_n\} \in S(T) \) such that

\[
\sum_{n=1}^{\infty} P(A_{t_n}) < \infty \quad \text{or} \quad \sum_{n=1}^{\infty} \left( 1 - P(A_{t_n}) \right) = \infty ,
\]

(iii) both (3.1) and (3.2) are satisfied if we have (3.3) for some \( \{t_n\} \in S(T) \) as well as (3.4) for some \( \{t'_n\} \in S(T) \).

Proof. a) If (3.3) is satisfied for some \( \{t_n\} \in S(T) \), then from the Borel-Cantelli Lemma we get \( P(\limsup_t A_{t_n}) = 1 \), i.e.

\[ P(\Omega \setminus \limsup_t A_{t_n}) = 0. \]

Since \( \limsup_t A_{t_n} \subset \bigcup_t A_t \), or equivalently \( \Omega \setminus \bigcup_t A_t \subset \Omega \setminus \limsup_t A_{t_n} \), one has \( \Omega \setminus \bigcup_t A_t \in \mathcal{A} \) and \( P(\Omega \setminus \bigcup_t A_t) = 0 \), by the completeness of the probability space.

Therefore (3.1) is valid.

b) If one of the conditions in (3.4) is satisfied for some \( \{t_n\} \in S(T) \), we have then

\[ \sum_{n=1}^{\infty} P(\overline{A}_{t_n}) = \infty . \]

Now (3.2) follows from (2.5) and the proof above for \( \overline{A}_t, t \in T \).

The following Theorem generalizes the 0—1 law of Kolmogorov.

**Theorem 2.** Let \( \{\sigma_t, t \in T\} \) with \( \text{card } T \geq \text{card } \mathbb{N} \) be a family of independent \( \sigma \)-algebras contained in \( \mathcal{A} \). Then

\[ P(A) = 0 \text{ or } 1 \quad \text{for all } A \in \bigcup_t \sigma_t. \]

Proof. Denote

\[ \mathcal{M} = \{ A : A \in \mathcal{A}, P(A) = 0 \text{ or } 1 \}. \]

The 0—1 law of Kolmogorov implies

\[ \mathcal{M} \supseteq \sigma_{\{t_n\}} \quad \text{for every } \{t_n\} \in S(T). \]

It follows from (3.6) that

\[ \mathcal{M} \supseteq \{ t_n \}. \]

Hence \( \mathcal{M} \) is an algebra containing the family \( \{ \sigma_{\{t_n\}}, \{t_n\} \in S(T) \} \). Moreover, \( \mathcal{M} \) is a monotone class. In fact, let \( \{ A_n \} \subset \mathcal{M}, A_n \uparrow \), then

\[ P(\lim_n A_n) = \lim_n P(A_n) = \begin{cases} 1 & \text{if there is } A_k \text{ such that } P(A) = 1, \\ 0 & \text{if } P(A_n) = 0 \quad \text{for all } n \in \mathbb{N}. \end{cases} \]
Hence \( \lim \uparrow A_n \in \mathcal{M} \). Similarly, one has also \( \lim \downarrow A_n \in \mathcal{M} \) for \( A_n \downarrow \) in \( \mathcal{M} \).
Therefore \( \mathcal{M} \) is a \( \sigma \)-algebra containing

\[
\sigma(\sigma(\{t_n\}, \{t_n\} \in S(T)) = \sup_T \sigma_t.
\]

This completes the proof.

References


Souhrn

ZÁKON 0—1 ZOBECNĚNÝ PRO NESPÔČETNÉ SYSTÉMY JEVŮ
A JEVOVÝCH \( \sigma \)-ALGEBER

Nguyen-van-Ho

Pojmy \( \lim \sup A_n \), \( \lim \inf A_n \) pro posloupnosti množin \( A_n \) a pojem \( \lim \sup \sigma_n \)
pro posloupnosti \( \sigma \)-algeber \( \sigma_n \) jsou v článku zobecněny pro nespočetné systémy
množin, resp. \( \sigma \)-algeber. Na základě těchto zobecněných definic se pak dokazuje
určitá slabší obdoba Borelova-Cantelliho lemmatu pro nespočetné systémy množin
\( A_n, t \in T \), a přímé zobecnění Kolmogorovova 0—1 zákona pro nespočetné systémy
\( \sigma \)-algeber \( \sigma_t, t \in T \).

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