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ON THE POSSIBILITY OF CALCULATION OF ZERO POINTS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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This paper deals with possibilities of calculating zero points of the differential equations

$$y'' + f(x)\, g(y) = p'(x) \tag{1}$$

$$Y'' + F(x)\, g(Y) = p'(x)\,. \tag{2}$$

The solutions of these equations may be expressed as

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \int_{x_0}^{x} [p(t) - p(x_0)]\, dt - \int_{x_0}^{x} (x - t)\, f(t)\, g[y(t)]\, dt \tag{3}$$

and

$$Y(x) = Y(x_0) + Y'(x_0)(x - x_0) + \int_{x_0}^{x} [p(t) - p(x_0)]\, dt - \int_{x_0}^{x} (x - t)\, F(t)\, g[Y(t)]\, dt\,, \tag{4}$$

respectively. Comparison theorems for $y(x)$ and $Y(x)$ can be used for approximate calculation of zero points of $y(x)$.

We shall assume that $f(x)$, $F(x)$ and $p'(x)$ are continuous on $I_x = \langle a; b\rangle$ where $-\infty < a < b < +\infty$ and that $g(y)$ is continuous on $I_y = \langle A; B\rangle$ where $-\infty < A < B < +\infty$; the intervals mentioned are such that $\forall x \in \langle x_0; b\rangle : y(x) \in \langle A; B\rangle$.

We shall further assume that the solutions of (1) and (2) exist and are unique on $\langle x_0; b\rangle$ for some $x_0 \in I_x$.

Theorem 1. *Suppose that* $y(x)$ *is a solution of* (1) *and* $Y(x)$ *a solution of* (2), $y(x_0) \geqq Y(x_0)$, $y'(x_0) \geqq Y'(x_0)$, *and that*

$$f(x_0)\frac{dg(y)}{dy} < 0 \ \forall y \in I_y , \tag{5}$$

$$[F(x) - f(x)]\, g(y) > 0 \ \forall (x, y) \in R = I_x x I_y , \tag{6}$$

$$f(x) \neq 0 , \quad F(x) \neq 0 \ \forall x \in I_x . \tag{7}$$

Then $\forall x \in (x_0; b\rangle : y(x) > Y(x) \wedge y'(x) > Y'(x) \wedge y''(x) > Y''(x)$.

Proof. Let $h(x) = y(x) - Y(x)$. By hypothesis

$$h''(x_0) = F(x_0)\, g[Y(x_0)] - f(x_0)\, g[y(x_0)] > 0 .$$

Thus there exists a right neighborhood 0_{x_0} of x_0 such that $\forall x \in 0_{x_0} : h(x) > 0$. The proof now will be indirect: Suppose that $\exists \bar{x} \in (x_0; b\rangle : h(\bar{x}) = 0$ but $\forall x \in (x_0; \bar{x}) : h(x) > 0$. Using (3) and (4), we obtain

$$0 = h(x_0) + h'(x_0)(x - x_0) + \int_{x_0}^{x} (\bar{x} - t)\{F(t)\, g[Y(t)] - f(t)\, g[y(t)]\}\, dt .$$

By hypothesis the right hand side of this equation is positive which yields a contradiction. This means that there exists no $\bar{x} \in (x_0; b\rangle$ with the above property and therefore $\forall x \in (x_0; b\rangle : h(x) > 0$. By hypothesis

$$F(x)\, g[Y(x)] - f(x)\, g[y(x)] > 0 \quad \text{if} \quad y(x) > Y(x) , \quad \text{i.e.}$$

$$\forall x \in (x_0; b\rangle : y'(x) > Y'(x) \wedge y''(x) > Y''(x) .$$

Corollary. *Let* $\min_{\langle x_0; b\rangle} f(x) = m$, $\max_{\langle x_0; b\rangle} f(x) = M$. *Let* $y_m(x)$ *and* $y_M(x)$ *be the solutions of* (1) *for* $f(x) \equiv m$ *and* $f(x) \equiv M$, *respectively. If* $\forall y \in I_y : g(y) < 0 \wedge y_m(x_0) \leqq y(x_0) \leqq y_M(x_0) \wedge y'_m(x_0) \leqq y'(x_0) \leqq y'_M(x_0)$ *then* $\forall x \in (x_0; b\rangle : y_m(x) < y(x) < y_M(x)$. *If* $\forall y \in I_y : g(y) > 0 \wedge y_M(x_0) \leqq y(x_0) \leqq y_m(x_0) \wedge y'_M(x_0) \leqq y'(x_0) \leqq y'_m(x_0)$ *then* $\forall x \in (x_0; b\rangle : y_M(x) < y(x) < y_m(x)$.

Let $u(x)$ be the solution of (1) with the initial conditions $u(x_0) = u_0 > y(x_0)$, $u'(x_0) = y'(x_0)$; then the following theorem holds.

Theorem 2. *If* $\forall (x, y) \in R = I_x \times I_y : f(x)\, dg(y)/dy < 0$ *then* $\forall x \in \langle x_0; b\rangle : y(x) < u(x)$.

Proof. Owing to (3) and to the initial conditions, $y(x) - u(x) = y(x_0) - u(x_0) + \int_{x_0}^{x} (x - t)\{g[u(t)] - g[y(t)]\}\, f(t)\, dt$.

Since $u(x_0) > y(x_0)$ there exists a right neighborhood 0_{x_0} of x_0 such that $\forall x \in 0_{x_0} : y(x) < u(x)$. Let $\bar{x} \in \langle x_0; b\rangle$ be a point such that $y(\bar{x}) = u(\bar{x}) \wedge \forall x \in \langle x_0; \bar{x}) : y(x) \neq u(x)$.

Then

$$0 = y(x_0) - u(x_0) + \int_{x_0}^{\bar{x}} (\bar{x} - t)\, f(t)\, \{g[u(t)] - g[y(t)]\}\, \mathrm{d}t$$

and the right hand side of this equation is always negative owing to the assumptions made; this again yields a contradiction and thus proves that $\forall x \in \langle x_0; b\rangle : y(x) < u(x)$.

If $y_\xi(x)$ is a solution of (1) for $f(x) \equiv f(\xi)$ satisfying the conditions $y_\xi(\xi) = 0$, $y'(x_0) = y'_\xi(x_0)$ and $y(x)$ is as stated in Theorem 1, then we have

Theorem 3. *Suppose that the functions $f(x)$, $g(y)$ satisfy the hypotheses of Theorem 1; suppose moreover that*

$$[y_\xi(x_0) - y(x_0)]\, f'(x_0) \frac{\mathrm{d}g(y)}{\mathrm{d}y} < 0 \quad \textit{for} \quad y = y_0 \quad \textit{and that} \quad \forall x \in \langle x_0; \xi\rangle : f'(x) \neq 0.$$

Then $\forall x \in (x_0; \xi) : [y(x_0) - y_\xi(x_0)]\,[y(x) - y_\xi(x)] > 0.$

Proof. We shall show that the theorem holds if $\forall x \in \langle x_0; b\rangle : f(x) > 0 \wedge \wedge f'(x_0) > 0$, $y_\xi(x_0) > y(x_0)$ and $g(y)$ is negative and decreasing. Let $Y(x)$ be a solution of (2) for $F(x) \equiv f(\xi)$ satisfying the conditions $Y(x_0) = y(x_0)$, $Y'(x_0) = y'(x_0)$. By Theorem 1 $\forall x \in (x_0; \xi) : y(x) < Y(x)$ and therefore by Theorem 2 $\forall x \in (x_0; \xi\rangle : : y_\xi(x) > Y(x) > y(x)$. In the other cases the proof is analogous.

The results obtained so far may be used for approximate calculation of zero points of solutions of (1).

Example. Calculate the zero point $\bar{x}$ of the solution $y(x)$ of the equation

$$y'' + e^{-2x}(1 - y) = 0 \tag{8}$$

with the initial conditions $y(0) = 1$, $y'(0) = -1$.
The auxiliary equation will be

$$Y'' + K_i^2(1 - Y) = 0 \tag{9}$$

and

$$Y_i(x) = 0{\cdot}5K_i^{-1}(e^{-K_i x} - e^{K_i x}) + 1 \tag{10}$$

is its solution under the initial conditions $Y(0) = 1$, $Y'(0) = -1$. If x_{i+1} is a zero of (10) then

$$x_{i+1} = K_i^{-1} \ln (K_i + \sqrt{(K_i^2 + 1)}) \tag{11}$$

or

$$x_{i+1} = \int_0^1 \frac{\mathrm{d}y}{\sqrt{(1 + K_i^2(y - 1)^2)}}\,. \tag{12}$$

To obtain an estimate of the interval $(a_1; b_1) \subset \langle x_0; b\rangle$ containing $\bar{x}$ we can use Theorem 1. As $f(x) = e^{-2x} \leqq M = 1$ for $x > 0$, $y_M(x) = 0{\cdot}5(e^{-x} - e^x) + 1$ is the solution of (9) for $K_i = 1$. For the zero point x_M of $y_M(x)$ we have $x_M > 0{\cdot}87$. Furthermore, $f(x) = e^{-2x} > 0$ for every $x \in \langle 0; \infty\rangle$. $y_m(x) = 1 - x$ is a solution of (9) for $K_i = 0$ and $x_m = 1$ is its zero point. Therefore $\bar{x} \in (0{\cdot}87; 1)$ by Theorem 1. Using the Romberg integration method, a computer calculated $x_1 = 0{\cdot}978714$, $x_2 = 0{\cdot}977842$, $x_3 = 0{\cdot}977805$ for $x_0 = b_1 = 1$. A similar result is obtained much more easily from (11). By Theorem 1, $\bar{x} < 0{\cdot}977805$.

Let $y_\xi(x)$ be the solution of (9) for $K_i = e^{-\xi}$ with the conditions $y_\xi(\xi) = 0$, $y'_\xi(0) = -1$ for $\xi = 0{\cdot}97780$. Calculation shows that $y_\xi(0) = 0{\cdot}999994 < 1$ and by Theorem 3 $\forall x \in (0; \xi) : y_\xi(x) < y(x)$. Thus we can see that $\exists \bar{x} \in (0{\cdot}97780; 0{\cdot}977805) : y(\bar{x}) = 0$.

Súhrn

O MOŽNOSTI VÝPOČTU NULOVÝCH BODOV RIEŠENÍ DIFERENCIÁLNYCH ROVNÍC DRUHÉHO RÁDU

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Práca sa zaoberá možnosťou výpočtu nulových bodov riešení diferenciálnych rovníc druhého rádu typu $y'' + f(x)\, g(y) = p'(x)$. Pomocou porovnávacích viet medzi riešeniami dvoch diferenciálnych rovníc druhého rádu je na príklade uvedený postup výpočtu nulového bodu.

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