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ALGORITMY

42. BISEC

THE EIGENVALUES OF THE SYMMETRIC EIGENPROBLEM $\mathbf{A}x = \lambda \mathbf{B}x$
AND RELATED EIGENPROBLEMS

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In some cases, the eigenvalues of a generalized eigenvalue problem

$$(1) \quad \mathbf{A}x = \lambda \mathbf{B}x,$$

where $\mathbf{A} = \{\mathbf{A}_{ij}\}_{i,j=1,\dots,n}$ and $\mathbf{B} = \{\mathbf{B}_{ij}\}_{i,j=1,\dots,n}$ are real symmetric matrices of order n , \mathbf{B} positive definite and of related eigenproblems

$$(2) \quad \mathbf{A}^{(k)}x = \lambda \mathbf{B}^{(k)}x, \quad k = 1, \dots, n - 1,$$

where

$$(3) \quad \mathbf{A}^{(k)} = \{\mathbf{A}_{ij}\}_{i,j=1,\dots,k} \quad \text{and} \quad \mathbf{B}^{(k)} = \{\mathbf{B}_{ij}\}_{i,j=1,\dots,k}$$

are submatrices of \mathbf{A} and \mathbf{B} , are required. An efficient method for calculating eigenvalues of all eigenproblems (1) and (2) is given here.

For a given real number μ , we decompose the matrix $\mathbf{A} - \mu\mathbf{B}$ as

$$(4) \quad \mathbf{A} - \mu\mathbf{B} = \mathbf{L} \cdot \mathbf{U},$$

where $\mathbf{L}(\mathbf{U})$ is the lower (upper) triangle matrix and $\mathbf{L}_{ii} = 1$. Then, the number of eigenvalues of the eigenproblem $\mathbf{A}^{(k)}x = \lambda \mathbf{B}^{(k)}x$ less than μ is equal to (for $\mathbf{B} = 1$ see e.g. [1]; for $\mathbf{B} \neq 1$, the generalization is straightforward)

$$(5) \quad I_k(\mu) = \sum_{i=1}^k \Theta(-\mathbf{U}_{ii}).$$

Here, $\Theta(x) = 1$ for $x > 0$, $\Theta(x) = 0$ otherwise. Making use of (5), we calculate the eigenvalues of (1) and (2) by the method of bisection (the procedure *bisec*). The

quantities $I_k(\mu)$ are calculated by the procedure *ik* (see also [2]). Some information obtained when determining one eigenvalue of (1) or (2) (for a given k) is, in general, of significance in the determination of other eigenvalues of the same eigenproblem (see [3]). In *bisec*, full advantage is taken of all relevant information and this results in a very substantial saving of time in case of a number of close or coincident eigenvalues. According to (5), the calculation of $I_k(\mu)$ gives the values of $I_1(\mu), \dots, I_{k-1}(\mu)$ as an additional result. The full use of this information further reduces the computational time.

The procedure *bisec* may be used to calculate the eigenvalues of all eigenproblems (1) and (2) in a given interval (e_{\min}, e_{\max}) . The matrices **A** and **B** are assumed to be band matrices ($a_{ij} = b_{ij} = 0$ for $|i - j| \geq m, m \leq n$). For the sake of efficiency, just the (i, j) elements ($0 \leq i - j < m$) of these matrices are stored, column by column, in one-dimensional arrays a and b . The number of arithmetic operations in one bisection step is about $1.5nm^2$. Therefore, the procedure *bisec* should not be used for band matrices with high half-bandwidth m . In general, it works most efficiently if the order n is very high, since the computational time is then roughly given by the time needed to calculate the eigenvalues of a few eigenproblems of the greatest order. The procedure *bisec* is in such cases much more efficient than other methods which calculate the eigenvalues of each eigenproblem separately. The accuracy of results is not influenced by close or coincident eigenvalues.

For large n , it is usually impossible to store all upper bounds e_{kl} (the upper bound to the l -th eigenvalue of the eigenproblem of the k -th order) or, respectively, lower bounds d_{kl} in the core. Hence, we store these two arrays (which are for the sake of simplicity assumed to be square arrays) on a disc row by row. The procedure *Read* ($k, d1, e1$) reads the k -th row of the arrays e and d from a disc and stores them in the arrays $e1$ and $d1$ in the core. Similarly, the procedure *Write* ($k, d1, e1$) stores the arrays $e1$ and $d1$ from the core in the k -th row of the arrays e and d on a disc, respectively.

procedure *ik*($a, b, mi, m, n, eps, rel, q$);

value mi, m, n, eps, rel ;

real mi, eps, rel ;

integer m, n ;

real array a, b ;

integer array q ;

comment Input to procedure *ik*

a, b $[(n - m + 1) \times m + (m - 1) \times m // 2] \times 1$ arrays giving the (i, j) elements ($0 \leq i - j < m$) of the matrices **A** and **B** stored column by column.

mi the number of eigenvalues less than mi will be found.

m bandwidth of **A** and **B** is $2m - 1$.

n order of **A** and **B**.

eps the smallest positive real number representable on the computer.
rel the smallest positive real number for which $1 + rel > 1$ on the computer.

Output of procedure *ik*

q $n \times 1$ array. $q[k]$ gives the number of eigenvalues less than m_i for the eigenproblem of the k -th order;

```

begin
  real c, x, ymax;
  integer i, i1, i2, j, k, l1, l2, n2;
  real array y[1 : m], w[1 : (n - m + 1) × m + (m - 1) × m//2];
  i2 := (n - m + 1) × m + (m - 1) × m//2;
  for i := 1 step 1 until i2 do w[i] := a[i] - mi × b[i];
  x := w[1];
  if x < eps then q[1] := 1 else q[1] := 0;
  i1 := 1;
  for i := 2 step 1 until n do
  begin n2 := n - i + 2;
    l1 := n2 - m;
    if l1 < 0 then l1 := 0;
    if m < n2 then n2 := m;
    ymax := 0;
    for j := 2 step 1 until n2 do
    begin i1 := i1 + 1;
      y[j] := c := w[i1];
      if abs(c) > ymax then ymax := abs(c)
    end j;
    if ymax ≥ eps then
    begin i2 := i1;
      for j := 2 step 1 until n2 do
      begin if abs(x) < eps then c := y[j]/ymax/rel
        else c := -y[j]/x;
        for k := j step 1 until n2 do
        begin i2 := i2 + 1;
          w[i2] := w[i2] + c × y[k]
        end k;
        l2 := j - 1;
        if l2 < l1 then i2 := i2 + l2 else i2 := i2 + l1
        end j
      end;
      i1 := i1 + 1;
      x := w[i1];
      if x < eps then q[i] := q[i - 1] + 1 else q[i] := q[i - 1];

```

```

end i
end ik;
procedure bisec (a, b, m, n, epsres, emin, emax, eps, rel) result: (m1, m2) exit: (fail);
value m, n, epsres, emin, emax, eps, rel;
real epsres, emin, emax, eps, rel;
integer m, n;
real array a, b;
integer array m1, m2;
label fail;
comment Input to procedure bisec
    a, b       $[(n - m + 1) \times m + (m - 1) \times m/2] \times 1$  arrays giving the
              (i, j) elements ( $0 \leq i - j < m$ ) of the matrices A and B
              stored column by column.
    m        bandwidth of A and B is  $2m - 1$ .
    n        order of A and B.
    epsres    the relative (for an eigenvalue  $> 1$ ) or the absolute (for an
              eigenvalue  $< 1$ ) error in any eigenvalue.
    emin, emax all eigenvalues in (emin, emax) will be computed.
    eps      the smallest positive real number representable on the com-
              puter.
    rel      the smallest positive real number for which  $1 + rel > 1$  on
              the computer.

    Output of procedure bisec
    m1, m2    $n \times 1$  arrays. For an eigenproblem of order k, the eigenvalues
              with sequential numbers  $m1[k], \dots, m2[k]$  lie in (emin, emax).

    e         $n \times n$  array stored on a disc gives the computed eigenvalues.
              The l-th eigenvalue of the k-th eigenproblem is stored as  $e_{kl}$ .
              For each eigenproblem, the eigenvalues are arranged in
              ascending order.

    fail     the exit used if B is not positive definite;
begin real g, h, mi; integer i, i1, i2, j, k, l, qk;
real array d1, e1, d2, e2[1 : n]; integer array q[1 : n];
ik(b, b, 0, m, n, eps, rel, q);
if  $q[n] > 0$  then go to fail;
comment The calculation of sequential numbers of the eigenvalues lying in (emin, emax);
ik(a, b, emin, m, n, eps, rel, m1);
ik(a, b, emax, m, n, eps, rel, m2);
for  $i := 1$  step 1 until  $n$  do
begin  $m1[i] := m1[i] + 1$ ;

```

```

    for j := m1[i] step 1 until m2[i] do
    begin d1[j] := emin;
        e1[j] := emax
    end j;
    Write(i, d1, e1)
end i;
for k := n step -1 until 1 do
begin
comment The calculation of the eigenvalues for the eigenproblem of the k-th order;
Read(k, d1, e1);
if m > k then m := k;
h := emax;
comment Loop for the l-th eigenvalue;
for l := m2[k] step -1 until m1[k] do
begin g := emin;
    for i := l step -1 until m1[k] do
    begin if g < d1[i] then
        begin g := d1[i];
            go to cont
        end
    end i;
cont: if h > e1[l] then h := e1[l];
comment The method of bisection;
for mi := (g + h)/2 while h - g > 2 × (epsres × abs(mi) + epsres) do
begin ik(a, b, mi, m, k, eps, rel, q);
    qk := q[k];
    if qk < l then
    begin if qk < m1[k] then d1[m1[k]] := g := mi
        else
        begin g := d1[qk + 1] := mi;
            if e1[qk] > mi then e1[qk] := mi
        end
    end
    else h := mi;
comment The calculation of new lower and upper bounds;
for i := k - 1 step -1 until 1 do
begin
Read(i, d2, e2);
for j := m1[i] step 1 until m2[i] do
if q[i] < j then
begin if d2[j] < mi then d2[j] := mi end
else

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        if e2[j] > mi then e2[j] := mi;
        Write(i, d2, e2)
    end i
end mi;
e1[l] := mi
end l;
comment New storage of a and b;
i1 := i2 := 0;
for i := 1 step 1 until k do
begin l := k - i + 1;
    if m < l then l := m;
    for j := i step 1 until i + l - 1 do
begin i1 := i1 + 1;
    if j < k then
begin i2 := i2 + 1;
        a[i2] := a[i1];
        b[i2] := b[i1]
    end
    end j
end i;
Write(k, d1, e1)
end k
end bisec;

```

Example.

For the data $a = (10, 2, 3, 12, 1, 2, 11, 1, 9)$,
 $b = (12, 1, -1, 14, 1, -1, 16, -1, 12)$, $m = 3$, $n = 4$, $epsres = 10^{-15}$,
 $emin = -10$, $emax = 10$, $eps = 10^{-75}$, $rel = 10^{-15}$ corresponding to the matrices

$$\mathbf{A} = \begin{pmatrix} 10 & 2 & 3 & 0 \\ 2 & 12 & 1 & 2 \\ 3 & 1 & 11 & 1 \\ 0 & 2 & 1 & 9 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 12 & 1 & -1 & 0 \\ 1 & 14 & 1 & -1 \\ -1 & 1 & 16 & -1 \\ 0 & -1 & -1 & 12 \end{pmatrix}$$

we obtained the following results (the computer ICL 4-72)

$$\begin{aligned}
 m1[1] &= m1[2] = m1[3] = m1[4] = 1, \\
 m2[1] &= 1, \quad m2[2] = 2, \quad m2[3] = 3, \quad m2[4] = 4, \\
 e[1, 1] &= 8.3333 \ 3333 \ 3333 \ 337_{10} - 1, \\
 e[2, 1] &= 7.4790 \ 6174 \ 4278 \ 959_{10} - 1, \\
 e[2, 2] &= 9.2874 \ 0532 \ 1589 \ 304_{10} - 1, \\
 e[3, 1] &= 4.9264 \ 3004 \ 8161 \ 612_{10} - 1, \\
 e[3, 2] &= 8.3439 \ 0032 \ 4405 \ 518_{10} - 1,
 \end{aligned}$$

$$\begin{aligned}
e[3, 3] &= 1.0765\ 2218\ 2107\ 887, \\
e[4, 1] &= 4.4739\ 1135\ 7782\ 800_{10} - 1, \\
e[4, 2] &= 6.5396\ 6400\ 2667\ 978_{10} - 1, \\
e[4, 3] &= 9.4074\ 1722\ 5080\ 661_{10} - 1, \\
e[4, 4] &= 1.1602\ 1950\ 8168\ 733.
\end{aligned}$$

References

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