Ivan Hlaváček

Dual finite element analysis for semi-coercive unilateral boundary value problems

Aplikace matematiky, Vol. 23 (1978), No. 1, 52–71

Persistent URL: http://dml.cz/dmlcz/103730

DUAL FINITE ELEMENT ANALYSIS FOR SEMI-COERCIVE UNILATERAL BOUNDARY VALUE PROBLEMS

IVAN HLAVÁČEK

(Received December 10, 1976)

A dual finite element procedure for unilateral coercive boundary value problems with homogeneous and inhomogeneous obstacles on the boundary has been presented in [1] and [2]. Some a priori error estimates have been shown provided the solutions were sufficiently regular. A posteriori error estimates and two-sided bounds for the energy follow from the duality approach.

In the present paper the dual analysis is extended to semi-coercive problems with homogeneous unilateral constraints on the boundary. Using the idea of Falk [7] and Mosco, Strang [3], analogous a priori error estimates are deduced, as previously. Moreover, the convergence of the finite element approximations to the primary problem is proved without any regularity assumption.

1. THE DUAL VARIATIONAL FORMULATIONS

Let us consider the following model problem

$$-\Delta u = f \quad \text{in} \quad \Omega \subset R^n\,, \tag{1.1}$$

$$u \geqq 0\,, \quad \frac{\partial u}{\partial \nu} \geqq 0\,, \quad u \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \equiv \partial\Omega\,, \tag{1.2}$$

where Ω is a bounded domain with Lipschitz boundary Γ, $f \in L_2(\Omega)$, $\partial u/\partial \nu$ is the derivative with respect to the outward normal to Γ.

We shall use the Sobolev spaces $H^k(\Omega)$ ($\equiv W^{k,2}(\Omega)$) with the usual norm $\|\cdot\|_k$, $H^0(\Omega) = L_2(\Omega)$ and denote $x = (x_1, \ldots, x_n)$,

$$(u, v)_0 = \int_\Omega uv \,\mathrm{d}x\,,$$

$$(\operatorname{grad} u, \operatorname{grad} v) = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \,\mathrm{d}x\,, \quad (u, v)_1 = (u, v)_0 + (\operatorname{grad} u, \operatorname{grad} v)\,.$$

$H^{1/2}(\Gamma)$ is the space of traces γv of functions $v \in H^1(\Omega)$ on the boundary Γ. We define the functional of potential energy

$$\mathscr{L}(v) = \tfrac{1}{2}|v|_1^2 - (f, v)_0\,,$$

where

$$|v|_1^2 = (\operatorname{grad} v, \operatorname{grad} v)\,,$$

and the convex cone

$$\mathscr{K} = \{v \in H^1(\Omega) \mid \gamma v \geqq 0 \text{ on } \Gamma\}\,.$$

Instead of the classical version (1.1), (1.2) of the problem, we introduce the following variational formulation:

to find $u \in \mathscr{K}$ such that

$$\mathscr{L}(u) \leqq \mathscr{L}(v) \quad \forall v \in \mathscr{K}\,. \tag{1.3}$$

The problem (1.3) will be called *primary*. It is easy to verify that any solution of (1.3) satisfies (1.1) in the sense of distributions and (1.2) in a functional sense. In fact, (1.3) is equivalent with

$$(\operatorname{grad} u, \operatorname{grad}(v - u)) \geqq (f, v - u)_0 \quad \forall v \in \mathscr{K}\,. \tag{1.4}$$

Inserting $v = u \pm \varphi$, $\varphi \in C_0^\infty(\Omega)$ (an infinitely smooth function with a compact support in Ω), we obtain (1.1) in the sense of distributions. Then the normal derivative $\partial u/\partial \nu$ represents a linear continuous functional on $H^{1/2}(\Gamma)$, if we define

$$\left\langle \frac{\partial u}{\partial \nu}, w \right\rangle = (\operatorname{grad} u, \operatorname{grad} v) - (f, v)_0\,, \quad \forall w \in H^{1/2}(\Gamma) \tag{1.5}$$

where $v \in H^1(\Omega)$ is such that $\gamma v \equiv w$.

Inserting $v = 0$ and $v = 2u$ into (1.4), we obtain, using also (1.5),

$$0 = (\operatorname{grad} u, \operatorname{grad} u) - (f, u)_0 = \left\langle \frac{\partial u}{\partial \nu}, u \right\rangle. \tag{1.6}$$

Then

$$0 \leqq (\operatorname{grad} u, \operatorname{grad} v) - (f, v)_0 = \left\langle \frac{\partial u}{\partial \nu}, \gamma v \right\rangle \quad \forall v \in \mathscr{K}\,. \tag{1.7}$$

The conditions (1.6) and (1.7) represent a weak form of the unilateral boundary conditions (1.2).

Conversely, if u is a classical (sufficiently smooth) solution of (1.1), (1.2), then multiplying (1.1) by a $v \in \mathscr{K}$ and integrating by parts, we derive (1.7) and (1.6), which in turn imply (1.4).

Lemma 1.1. *Assume that*

$$(f, 1)_0 < 0\,. \tag{1.8}$$

Then there exists a unique solution of the primary problem (1.3). *The solution of* (1.3) *exists only if*

$$(f, 1)_0 \leqq 0\,. \tag{1.9}$$

Proof. (cf. [4] – chpt. 1) 1. *Existence.* Let $\Gamma_0 \subset \Gamma$ be any open part of the boundary with positive measure. Define

$$\bar{v} = (\text{mes}\,\Gamma_0)^{-1} \int_{\Gamma_0} \gamma v \,\mathrm{d}\Gamma \quad \forall v \in H^1(\Omega)\,.$$

Then for

$$\tilde{v} = v - \bar{v}$$

we have

$$\int_{\Gamma_0} \gamma\tilde{v}\,\mathrm{d}\Gamma = 0\,,$$

$$|\tilde{v}|_1 \geqq C\|\tilde{v}\|_1$$

We may write

$$\mathscr{L}(v) = \tfrac{1}{2}|\tilde{v}|_1^2 - (f, \tilde{v})_0 - \bar{v}(f, 1)_0 \geqq \tfrac{1}{2}C^2\|\tilde{v}\|_1^2 - c_1\|\tilde{v}\|_1 - \bar{v}(f, 1)_0\,.$$

If $v \in \mathscr{K}$, $\|v\|_1 \to \infty$, then at least one of the norms $\|\tilde{v}\|_1$ and $\|\bar{v}\|_1 = \bar{v}(\text{mes}\,\Omega)^{1/2}$ goes to infinity. Hence (1.8) implies $\mathscr{L}(v) \to +\infty$, $\mathscr{L}$ is coercive over $\mathscr{K}$. As the set $\mathscr{K}$ is convex and closed in $H^1(\Omega)$, the minimizing element exists.

2. *Uniqueness.* Let u' and u'' be two solutions. Inserting them into (1.4) and subtracting, we obtain

$$|u'' - u'|_1^2 \leqq 0\,,$$

consequently, $u'' - u' = \text{const.}$ Denote $u'' = u$, $u' = u + c$ and suppose that $c \neq 0$. We have

$$\mathscr{L}(u) = \mathscr{L}(u + c) \Rightarrow (f, u)_0 = (f, u + c)_0 \Rightarrow (f, 1)_0 = 0\,,$$

which contradicts (1.8). Hence $c = 0$.

3. Let $a \in R^1$, $a \to +\infty$. Obviously, for $v_0 \equiv a$ $v_0 \in \mathscr{K}$. If a solution of (1.3) exists, $\mathscr{L}(v_0)$ is bounded below,

$$\lim_{a \to +\infty} \mathscr{L}(v_0) = -\lim a(f, 1)_0 > -\infty$$

and (1.9) follows.

For completeness, we discuss also the case, when the mean value of f vanishes.

Lemma 1.2 ([4] – chpt. 1). *Assume that*

$$(f, 1)_0 = 0 \tag{1.10}$$

Let $w \in H^1(\Omega)$ be a weak solution to the following Neumann's problem

$$-\Delta w = f \ in \ \Omega\,, \quad \partial w/\partial v = 0 \ on \ \Gamma\,, \quad \int_{\Gamma_0} \gamma w \, d\Gamma = 0\,, \tag{1.11}$$

where Γ_0 is any open part (non-empty) of Γ. Then the primary problem (1.3) has a solution, if and only if γw is bounded below on Γ. If this condition is satisfied, then all solutions possess the form $u = w + c$, where c is any constant such that $\gamma w + c \geqq 0$ on Γ.

Remark 1.1. For a smooth boundary $w \in H^2(\Omega)$, consequently $\gamma w \in H^{3/2}(\Gamma)$, which implies $\gamma w \in C(\Gamma)$, provided the space dimension $n \leqq 3$. The same assertion is true for convex polygonal domains in R^2.

Proof of lemma 1.2. Let (1.10) hold. From (1.5) we obtain

$$0 = (f, 1)_0 = -\left\langle \frac{\partial u}{\partial v}, 1 \right\rangle .$$

As the function $v_0 \equiv 1$ belongs to $\mathscr{K}$ and $\partial u/\partial v \geqq 0$, the condition $\partial u/\partial v = 0$ on Γ follows. By comparison with the problem (1.11) we deduce that $u = w + c$, where c is such that $\gamma w + c \geqq 0$ on Γ.

Next we shall introduce the *dual variational formulation*. To this end, we define

$$Q = \{\boldsymbol{q} \in [L_2(\Omega)]^n, \ \operatorname{div} \boldsymbol{q} \equiv \sum_{i=1}^{n} \partial q_i/\partial x_i \in L_2(\Omega)\}\,,$$

where the operator div is defined in the sense of distributions:

$$\int_\Omega \boldsymbol{q} \,.\, \operatorname{grad} \varphi \, dx = -\int_\Omega \varphi \operatorname{div} \boldsymbol{q} \, dx \quad \forall \varphi \in C_0^\infty(\Omega)\,.$$

For $\boldsymbol{q} \in Q$ the linear continuous functional $\boldsymbol{q} \,.\, v \in H^{-1/2}(\Gamma)$ can be defined by means of the relation

$$\langle \boldsymbol{q} \,.\, v, w \rangle = \int_\Omega (\boldsymbol{q} \,.\, \operatorname{grad} v + v \operatorname{div} \boldsymbol{q}) \, dx \quad \forall w \in H^{1/2}(\Gamma)\,,$$

where $v \in H^1(\Omega)$ is such that $\gamma v = w$.

We write $\boldsymbol{q} \,.\, v|_\Gamma \geqq 0$ if

$$\langle \boldsymbol{q} \,.\, v, s \rangle \geqq 0 \quad \forall s \in H^{1/2}(\Gamma)\,, \quad s \geqq 0\,.$$

Let us introduce the set of admissible functions

$$\mathscr{U} = \{\boldsymbol{q} \in Q \mid \operatorname{div} \boldsymbol{q} + f = 0\,, \quad \boldsymbol{q} \,.\, \nu|_\Gamma \geqq 0\}$$

and the functional (complementary energy)

$$\mathscr{S}(\boldsymbol{q}) = \tfrac{1}{2} \sum_{i=1}^{n} \|q_i\|_0^2\,.$$

The problem to find $\lambda^0 \in \mathscr{U}$ such that

$$\mathscr{S}(\lambda^0) \leqq \mathscr{S}(\boldsymbol{q}) \quad \forall \boldsymbol{q} \in \mathscr{U} \tag{1.12}$$

will be called *dual* to the primary problem (1.3).

Lemma 1.3. *The dual problem has a solution if and only if*

$$(f, 1) \leqq 0\,. \tag{1.13}$$

If (1.13) *holds, the solution is unique.*

Proof. The condition (1.13) is necessary and sufficient for the set $\mathscr{U}$ to be non-empty. In fact, let a $\boldsymbol{q} \in \mathscr{U}$ exist. As $v_0 \equiv 1$ belongs to $\mathscr{K}$, we have

$$0 \leqq \langle \boldsymbol{q} \,.\, \nu, 1 \rangle = (\operatorname{div} \boldsymbol{q}, 1)_0 = -(f, 1)_0\,,$$

consequently, (1.13) is necessary.

Conversely, let (1.13) be satisfied. Consider any solution of the Neumann's problem

$$-\Delta w = f \text{ in } \Omega\,, \qquad \partial w/\partial \nu = k \text{ on } \Gamma\,,$$

where

$$k = -(f, 1)_0/\operatorname{mes} \Gamma\,.[1]$$

As $k \geqq 0$, $\boldsymbol{q} = \operatorname{grad} w$ belongs to $\mathscr{U}$ and the set is non-empty.

$\mathscr{U}$ is closed and convex, the functional $\mathscr{S}$ strictly convex in $[L_2(\Omega)]^n$ and continuously differentiable. Hence the existence and uniqueness of the minimizing element follows.

Theorem 1.1. Let (1.13) hold and the primary problem have a solution u (or solutions $u + c$ in case of Lemma 1.2). Then the solution λ^0 of the dual problem satisfies the following relations

$$\lambda^0 = \operatorname{grad} u\,, \tag{1.14}$$

$$\mathscr{S}(\lambda^0) + \mathscr{L}(u) = 0\,. \tag{1.15}$$

---

[1]) Then the solution exists, because

$$\int_\Gamma k \,\mathrm{d}\Gamma + \int_\Omega f \,\mathrm{d}x = 0\,.$$

Proof. Let us introduce new parameters

$$\mathcal{N}_i = \frac{\partial v}{\partial x_i}, \quad i = 1, 2, \ldots, n$$

as contraints and the notation

$$M = [L_2(\Omega)]^n, \quad \mathcal{W} = \mathcal{K} \times M .$$

Then we may write

$$\text{(1.16)} \qquad \operatorname*{Inf}_{v \in \mathcal{K}} \mathcal{L}(v) = \operatorname*{Inf}_{[v, \mathcal{N}] \in \mathcal{W}} \operatorname*{Sup}_{\mu \in M} \mathcal{H}(v, \mathcal{N}; \mu) ,$$

where

$$\mathcal{H}(v, \mathcal{N}; \mu) = \tfrac{1}{2} \sum_{i=1}^{n} \|\mathcal{N}_i\|_0^2 - (f, v)_0 + \sum_{i=1}^{n} \left(\mu_i, \frac{\partial v}{\partial x_i} - \mathcal{N}_i\right)_0 .$$

In fact

$$\operatorname*{Sup}_{\mu \in M} \sum_{i=1}^{n} \left(\mu_i, \frac{\partial v}{\partial x_i} - \mathcal{N}_i\right)_0 = \begin{cases} 0 & \text{if } \mathcal{N} = \operatorname{grad} v \\ +\infty & \text{if } \exists i, \ \mathcal{N}_i \neq (\partial v/\partial x_i) \end{cases}$$

and consequently,

$$\operatorname*{Inf}_{[v, \mathcal{N}] \in \mathcal{W}} \operatorname*{Sup}_{\mu \in M} \mathcal{H}(v, \mathcal{N}; \mu) = \operatorname*{Inf}_{[v, \mathcal{N}] \in \mathcal{W}, \mathcal{N} = \operatorname{grad} v} \mathcal{H}(v, \mathcal{N}; \mu) = \operatorname*{Inf}_{v \in \mathcal{K}} \mathcal{L}(v) .$$

Let us investigate the problem dual to the problem (1.16), i.e.,

$$\operatorname*{Sup}_{\mu \in M} \operatorname*{Inf}_{[v, \mathcal{N}] \in \mathcal{W}} \mathcal{H}(v, \mathcal{N}; \mu) .$$

First of all we may write

$$-S(\mu) \equiv \operatorname*{Inf}_{[v, \mathcal{N}] \in \mathcal{W}} \mathcal{H}(v, \mathcal{N}; \mu) \leqq \operatorname*{Inf}_{[v, \mathcal{N}] \in \mathcal{W}, \mathcal{N} = \operatorname{grad} v} \mathcal{H}(v, \mathcal{N}; \mu) =$$

$$= \operatorname*{Inf}_{v \in \mathcal{K}} \mathcal{L}(v) = \mathcal{L}(u) \quad \forall \mu \in M ,$$

consequently

$$\text{(1.17)} \qquad \operatorname*{Sup}_{\mu \in M} [-S(\mu)] \leqq \mathcal{L}(u) .$$

On the other hand,

$$\text{(1.18)} \qquad -S(\mu) = \operatorname*{Inf}_{[v, \mathcal{N}] \in \mathcal{W}} \{\mathcal{H}_1(\mathcal{N}, \mu) + \mathcal{H}_2(v, \mu)\} ,$$

where

$$\mathcal{H}_1(\mathcal{N}, \mu) = \tfrac{1}{2} \sum_{i=1}^{n} \|\mathcal{N}_i\|_0^2 - \sum_{i=1}^{n} (\mu_i, \mathcal{N}_i)_0 , \quad \mathcal{H}_2(v, \mu) = -(f, v)_0 + \sum_{i=1}^{n} \left(\mu_i, \frac{\partial v}{\partial x_i}\right)_0 .$$

It is readily seen that the infimum of $\mathscr{H}_1$ is attained precisely if $\mathscr{N}_i = \mu_i \; \forall i = 1, \ldots, n$ and

$$(1.19) \qquad \operatorname*{Inf}_{\mathscr{N} \in M} \mathscr{H}_1(\mathscr{N}, \mu) = -\tfrac{1}{2} \sum_{i=1}^{n} \|\mu_i\|_0^2 .$$

Next we can show that

$$(1.20) \qquad \operatorname*{Inf}_{v \in \mathscr{K}} \mathscr{H}_2(v, \mu) = \begin{cases} 0 & \text{if } \mu \in \mathscr{U} , \\ -\infty & \text{if } \mu \in M \dot{-} \mathscr{U} . \end{cases}$$

In fact, $\mathscr{H}_2(v, \mu)$ is a linear continuous functional on $H^1(\Omega)$. Let a $v_0 \in \mathscr{K}$ exist such that $\mathscr{H}_2(v_0, \mu) < 0$. Then the infimum is $-\infty$, as $\mathscr{H}_2(tv_0, \mu) \to -\infty$ for $t \to +\infty$. For the infimum to be finite, it is therefore necessary that

$$(1.21) \qquad \mathscr{H}_2(v, \mu) \geqq 0 \quad \forall v \in \mathscr{K} .$$

Choosing $v = \pm\varphi$, $\varphi \in C_0^\infty(\Omega) \subset \mathscr{K}$, we obtain

$$0 = \mathscr{H}_2(\varphi, \mu) = -(f, \varphi)_0 + \sum_{i=1}^{n} \left(\mu_i, \frac{\partial \varphi}{\partial x_i}\right)_0 \quad \forall \varphi \in C_0^\infty(\Omega) .$$

Hence $\mu \in Q$, $f + \operatorname{div} \mu = 0$ and we may write

$$\sum_{i=1}^{n} \left(\mu_i, \frac{\partial v}{\partial x_i}\right)_0 = -(\operatorname{div} \mu, v)_0 + \langle \mu \,.\, \nu, \gamma v \rangle = (f, v)_0 + \langle \mu \,.\, \nu, \gamma v \rangle$$

for all $v \in H^1(\Omega)$. Hence we obtain

$$(1.22) \qquad \mathscr{H}_2(v, \mu) = \langle \mu \,.\, \nu, \gamma v \rangle \quad \forall v \in H^1(\Omega) .$$

Inserting (1.22) into (1.21), we conclude that $\mu \,.\, \nu|_\Gamma \geqq 0$. Altogether the infimum in (1.20) is bounded only if $\mu \in \mathscr{U}$. Conversely, if $\mu \in \mathscr{U}$, then (1.22) and (1.21) hold, which lead to (1.20).

Next from (1.18), (1.19) and (1.20) it follows

$$-S(\mu) = \begin{cases} -\tfrac{1}{2} \sum_{i=1}^{n} \|\mu_i\|_0^2 = -\mathscr{S}(\mu) & \forall \mu \in \mathscr{U} , \\ -\infty & \forall \mu \in M \dot{-} \mathscr{U} . \end{cases}$$

Finally, we have

$$(1.23) \qquad \operatorname*{Sup}_{\mu \in M} [-S(\mu)] = \operatorname*{Sup}_{\mu \in \mathscr{U}} [-\mathscr{S}(\mu)] = -\operatorname*{Inf}_{\mu \in \mathscr{U}} \mathscr{S}(\mu) = -\mathscr{S}(q^0) .$$

Let us set $\hat{\mathbf{q}} = \operatorname{grad} u$ and show that $\hat{\mathbf{q}} = \lambda^0$. In fact, $|u|_1^2 = (f, u)_0$ (see (1.6)) and therefore

$$\mathscr{L}(u) = -\tfrac{1}{2}|u|_1^2 = -\tfrac{1}{2} \sum_{i=1}^{n} \|\hat{\mathbf{q}}_i\|_0^2 = -\mathscr{S}(\hat{\mathbf{q}}) .$$

Moreover, $\hat{\mathbf{q}} \in \mathscr{U}$ as $\hat{\mathbf{q}} \cdot \nu|_\Gamma = \left.\frac{\partial u}{\partial \nu}\right|_\Gamma \geqq 0$ by virtue of (1.7). Consequently,

$$\operatorname*{Sup}_{\mu \in \mathscr{U}} [-\mathscr{S}(\mu)] \geqq -\mathscr{S}(\hat{\mathbf{q}}) = \mathscr{L}(u) .$$

With regard to (1.23) and (1.17) the equality holds, i.e.,

$$-\mathscr{S}(\lambda^0) = -\inf_{\mu \in \mathscr{U}} \mathscr{S}(\mu) = \mathscr{L}(u) = \operatorname*{Inf}_{v \in \mathscr{K}} \mathscr{L}(v) .$$

The uniqueness of the solution of the dual problem implies that $\hat{\mathbf{q}} = \lambda^0$.

2. FINITE ELEMENT APPROXIMATIONS TO THE PRIMARY PROBLEM

Assume that $\Omega \subset R^2$ is a bounded polygonal domain and let[1])

$$(2.1) \qquad (f, 1)_0 < 0$$

We carve Ω into triangles T generating a triangulation $\mathscr{T}_h$. Denote h the maximal side of all triangles in $\mathscr{T}_h$ and let V_h be the space of continuous piecewise linear functions on the triangulation $\mathscr{T}_h$.

We say that a *family of triangulations* $\{\mathscr{T}_h\}$, $0 < h \leqq 1$, *is α-β-regular*, if there exist positive α and β, such that for any h (i) the minimal angle of all triangles is not less than α and (ii) the ratio between any two sides in $\mathscr{T}_h$ is less than β.

Let us introduce the set

$$\mathscr{K}_h = V_h \cap \mathscr{K} = \{v \in V_h \mid v \geqq 0 \text{ on } \Gamma\} .$$

We say that $u_h \in \mathscr{K}_h$ is a *finite element approximation to the primary problem* if

$$(2.2) \qquad \mathscr{L}(u_h) \leqq \mathscr{L}(v) \quad \forall v \in \mathscr{K}_h .$$

There exists a unique solution of (2.2). In fact, $\mathscr{K}_h$ is closed and convex subset of $H^1(\Omega)$. $\mathscr{L}$ is coercive on $\mathscr{K}$ (see the proof of Lemma 1.1), consequently, it is coercive on $\mathscr{K}_h$, as well. As $\mathscr{L}$ is convex and differentiable, the solution u_h exists.

The uniqueness can be proved by the same argument as in Lemma 1.1.

To find u_h, we may employ e.g. the procedure of Gauss-Seidel with constraints (cf. [6] – chpt. 4 or [1]). Thus we obtain a sequence of iterations $v^m \in \mathscr{K}_h$, which converges to u_h for $m \to \infty$.

Next we shall estimate the distance between the solution u of the primary problem (1.3) and the finite element approximation u_h. To this end we employ a modified approach by Falk [7], which is based on the following

[1]) If $(f, 1)_0 = 0$, Theorem 1.1 and Lemma 1.2 yield that we can solve the classical Neumann problem and its dual formulation (cf. [5]).

Lemma 2.1. *It holds*

$$(2.3) \qquad |u - u_h|_1^2 \leqq (f, u - v_h)_0 + (\text{grad}\, u, \text{grad}\,(v_h - u)) + $$
$$+ (\text{grad}\,(u_h - u), \text{grad}\,(v_h - u)) \quad \forall v_h \in \mathscr{K}_h .$$

Proof. Inserting $v \equiv u_h$ into (1.4), we obtain

$$|u|_1^2 \leqq (\text{grad}\, u, \text{grad}\, u_h) + (f, u - u_h)_0 .$$

By a similar argument, we deduce

$$|u_h|_1^2 \leqq (\text{grad}\, u_h, \text{grad}\, v_h) + (f, u_h - v_h)_0 .$$

Then we may write

$$|u - u_h|_1^2 = |u|_1^2 + |u_h|_1^2 - 2(\text{grad}\, u, \text{grad}\, u_h) \leqq$$
$$\leqq (f, u - v_h)_0 + (\text{grad}\, u, \text{grad}\, u_h) + (\text{grad}\, u_h, \text{grad}\, v_h) - 2(\text{grad}\, u, \text{grad}\, u_h) =$$
$$= (f, u - v_h)_0 + (\text{grad}\, u, \text{grad}\,(v_h - u)) + (\text{grad}\,(u_h - u), \text{grad}\,(v_h - u)) .$$

Theorem 2.1. *Let $u \in H^2(\Omega)$ and $\gamma u \in H^2(\Gamma_m)$ for any side Γ_m, $m = 1, 2, \ldots, G$ of the polygonal boundary Γ. Then it holds*

$$(2.4) \qquad |u - u_h|_1 \leqq Ch\{\|u\|_2 + \sum_{m=1}^{G} \|u\|_{H^2(\Gamma_m)}\}$$

where C is independent of h and u.

Proof. Integrating by parts, we obtain

$$(\text{grad}\, u, \text{grad}\,(v_h - u)) + (f, u - v_h)_0 =$$
$$= (-\Delta u, v_h - u)_0 + \int_\Gamma \frac{\partial u}{\partial \nu}(v_h - u)\, \mathrm{d}s + (f, u - v_h)_0 = \int_\Gamma \frac{\partial u}{\partial \nu}(v_h - u)\, \mathrm{d}s .$$

From (2.3) it follows for any $v_h \in \mathscr{K}_h$

$$(2.5) \qquad |u - u_h|_1^2 \leqq \tfrac{1}{2}|u_h - u|_1^2 + \tfrac{1}{2}|v_h - u|_1^2 + \left\|\frac{\partial u}{\partial \nu}\right\|_{L_2(\Gamma)} \|v_h - u\|_{L_2(\Gamma)} .$$

Let us insert $v_h = u_I$, i.e. the Lagrange linear interpolate of u with the nodes given by $\mathscr{T}_h$. Then it holds

$$(2.6) \quad |u_I - u|_1 \leqq Ch\|u\|_2 , \quad \|u_I - u\|_{L_2(\Gamma_m)} \leqq Ch^2\|u\|_{H^2(\Gamma_m)} , \quad \left\|\frac{\partial u}{\partial \nu}\right\|_{L_2(\Gamma)} \leqq C\|u\|_2$$

and the assertion (2.4) follows from (2.5), (2.6).

Lemma 2.2. *Let* (2.1) *hold. Then*

$$c_h = \min_{x \in \Gamma} u_h(x) = 0 . \tag{2.7}$$

Proof. Assume that $c_h > 0$ and set $\hat{u}_h = u_h - c_h$. Then

$$\hat{u}_h \in \mathscr{K}_h , \quad \mathscr{L}(\hat{u}_h) = \mathscr{L}(u_h) + c_h(f, 1)_0 < \mathscr{L}(u_h) ,$$

which is a contradiction.

Remark 2.1. According to (2.4) we conclude that

$$\inf_{c \in R^1} \|u_h + c - u\|_1 = 0(h) . \tag{2.8}$$

With respect to (2.7), if the "optimal" constant in (2.8) $c \neq 0$, the minimum of $u_h + c$ over Γ differs from zero. It is well-known, however, (cf. [4]) that the trace γu vanishes on a set of positive measure. Therefore the violation of (2.7) may be unsuitable. Consequently, we are satisfied by u_h itself.

3. CONVERGENCE OF THE FINITE ELEMENT APPROXIMATIONS WITHOUT ANY REGULARITY ASSUMPTION

The a priori estimate (2.4) has been obtained under strong regularity assumptions. In general, however, such a regularity cannot be expected for domains with angular boundary points (cf. [8]). Therefore we have to study the convergence of u_h to a general $u \in \mathscr{K}$. To this end, we employ the following abstract theorem.

Theorem 3.1 (cf. [6] – chpt. 4). *Let V be a Hilbert space with the norm $\|\cdot\|$ and a seminorm $|\cdot|$, $\mathscr{K} \subset V$ a closed convex subset, $h \in (0, 1\rangle$ a real parameter, $\mathscr{K}_h \subset \mathscr{K}$ convex closed sets for any h.*

Let a differentiable functional $\mathscr{J}$ on V be given which is coercive on $\mathscr{K}$, the second differential (in the sense of Gâteaux) exists and satisfies the following inequalities

$$\alpha_0 |z|^2 \leqq D^2 \mathscr{J}(u; z, z) \leqq C\|z\|^2 \quad \forall u \in \mathscr{K} , \quad z \in V . \tag{3.1}$$

Denote u and u_h the minimizing elements of $\mathscr{J}$ over the sets $\mathscr{K}$ and $\mathscr{K}_h$, respectively. Let them be unique. Assume that $v_h \in \mathscr{K}_h$ exist such that

$$\lim \|u - v_h\| = 0 \quad for \quad h \to 0 . \tag{3.2}$$

Then it holds

$$\lim |u - u_h| = 0 \quad for \quad h \to 0 . \tag{3.3}$$

Proof. From (3.1) and the coerciveness of $\mathscr{J}$ the existence of u and u_h follows. Let $v_h \in \mathscr{K}$ satisfy (3.2). Using the Taylor's theorem we may write

$$\mathscr{J}(v_h) = \mathscr{J}(u) + D\,\mathscr{J}(u, v_h - u) + \tfrac{1}{2} D^2\,\mathscr{J}(u + \vartheta_h(v_h - u); v_h - u, v_h - u) .$$

By virtue of (3.1), we conclude

$$\lim \mathscr{J}(v_h) = \mathscr{J}(u) . \tag{3.4}$$

From the definition of u_h it follows

$$\mathscr{J}(u_h) \leqq \mathscr{J}(v_h) , \tag{3.5}$$

consequently,

$$\mathscr{J}(u_h) \leqq C < +\infty \ \forall h .$$

Since $\mathscr{J}$ is coercive on $\mathscr{K}$ and $u_h \in \mathscr{K}_h \subset \mathscr{K}$,

$$\|u_h\| \leqq C_1 < +\infty \ \forall h$$

and we can choose a subsequence (denote it again by $\{u_h\}$), such that $u_h \in \mathscr{K}_h$, u_h tends to u^* weakly. As $\mathscr{K}$ is weakly closed, $u^* \in \mathscr{K}$. We have

$$\mathscr{J}(u^*) \leqq \lim \mathscr{J}(u_h) = \mathscr{J}(u) ,$$

consequently, $u^* = u$.

There exist $\lambda_h \in (0, 1)$ such that

$$\mathscr{J}(u_h) = \mathscr{J}(u) + D\,\mathscr{J}(u; u_h - u) + \tfrac{1}{2}D^2\,\mathscr{J}(u + \lambda_h(u_h - u);\ u_h - u,\ u_h - u)$$

and by virtue of (3.1)

$$\mathscr{J}(u_h) - \mathscr{J}(u) - D\,\mathscr{J}(u, u_h - u) \geqq \tfrac{1}{2}\alpha_0 |u_h - u|^2 .$$

From (3.4), (3.5) and the weak convergence $u_h \rightharpoonup u$, the assertion (3.3) follows for the subsequence. Since the solution u is unique, the whole sequence satisfies (3.3).

Q.E.D.

Setting $\mathscr{J} = \mathscr{L}$, $V = W^{1,2}(\Omega)$, and assuming (2.1), we have the coerciveness of $\mathscr{J}$ over $\mathscr{K}$, (see the proof of Lemma 1.1) and (3.1) is satisfied for $|z| \equiv |z|_1$ with $\alpha_0 = c = 1$. It remains to verify (3.2).

Lemma 3.1. The set

$$\mathscr{K} \cap C^\infty(\bar{\Omega})$$

is dense in $\mathscr{K}$.

Proof. Let $u \in \mathscr{K}$ be any fixed function. There exists a function $v \in H^1(\Omega)$ such that $\gamma v = \gamma u$ on Γ and $v \geqq 0$ in Ω (see [9] – chpt. 2. Th. 5.7). Then

$$u = v + z ,$$

where $z \in H_0^1(\Omega)$ can be approximated by functions from $C_0^\infty(\Omega) \subset \mathscr{K}$. Hence it suffices to find a suitable approximation of v. To this end we extend v as follows.

Let the system $\{B_i\}$, $i = 0, 1, \ldots, r$ of open domains cover $\bar{\Omega}$ and $\{\varphi_i\}$ be the corresponding partition of unity, (i.e., $\varphi_i \in C_0^\infty(B_i)$, $0 \leqq \varphi_i \leqq 1$, $\sum_{i=0}^{r} \varphi_i(x) = 1$ $\forall x \in \bar{\Omega}$). Let $\bar{B}_0 \subset \Omega$ and $\bigcup_{i=1}^{r} B_i$ cover the boundary Γ. Denoting $v_j = v\varphi_j$, we have

$$v = \sum_{j=0}^{r} v_j\,, \quad v_j \in H^1(\Omega)\,, \quad \operatorname{supp} v_j \in B_j \ \forall j\,.$$

Consider any fixed v_j in B_j. We map $B_j \cap \bar{\Omega}$ into the upper halfplane $\{(\xi, \eta) |\ \eta \geqq 0\}$ by means of the mapping

$$\left.\begin{aligned} \xi &= x_1 \\ \eta &= x_2 - a(x_1) \end{aligned}\right\} \equiv (\xi, \eta) = T[(x_1, x_2)]$$

where $x_2 = a(x_1)$ represents the "angle" $B_j \cap \Gamma$. Then defining $\hat{v}_j(\xi, \eta) \equiv v_j(\xi, \eta + a(\xi))$, we have $\hat{v} \in H^1(\hat{B}_{j0})$, where $\hat{B}_{j0} = T(B_j \cap \Omega)$. The extension $P\hat{v}_j$ will be defined through

$$P\hat{v}_j(\xi, +\eta) = P\hat{v}_j(\xi, -\eta)\,.$$

Finally, we define

$$P\hat{v}_j(x_1, x_2 - a(x_1)) = Pv_j(x_1, x_2)\,.$$

Then $Pv_j \in H^1(B_j)$.

Let us consider the regularized function

$$R_\varkappa Pv_j(x) = \int_{B_j} \omega(x - x', \varkappa)\, Pv_j(x')\, \mathrm{d}x'\,, \quad x' \equiv (x_1', x_2')\,,$$

where

$$\omega(x, \varkappa) = \begin{cases} A\, \varkappa^{-2} \exp\left(\dfrac{|x|^2}{|x|^2 - \varkappa^2}\right), & \text{for } |x| < \varkappa\,, \\ 0 \quad \text{for } |x| \geqq \varkappa\,, \end{cases}$$

A and $\varkappa$ are positive constants, $x \equiv (x_1, x_2)$. As $Pv_j \geqq 0$ and $\omega \geqq 0$, we have

$$v_{j\varkappa} \equiv R_\varkappa Pv_j \geqq 0 \quad \forall x \in \Gamma\,,$$

$v_{j\varkappa} \in C^\infty(\bar{\Omega})$ and $\|v_{j\varkappa} - v_j\|_1 \to 0$ for $\varkappa \to 0$. For B_0, $v_0 \in H_0^1(B_0)$ will be approximated by a $v_{0\varkappa} \in C_0^\infty(B_0)$. Setting

$$v_\varkappa = \sum_{j=0}^{r} v_{j\varkappa}\,,$$

we obtain

$$\|v_\varkappa - v\|_1 \leqq \sum_{j=0}^{r} \|v_{j\varkappa} - v_j\|_1 \to 0 \quad \text{for} \quad \varkappa \to 0\,,$$

$v_\varkappa \in C^\infty(\bar{\Omega})$, $v_\varkappa \geqq 0$ on Γ. The proof is complete.

Theorem 3.2. *The finite element approximations converge "in the seminorm" to the solution u, i.e.*

$$\lim |u - u_h|_1 \to 0 \quad \text{for} \quad h \to 0 . \tag{3.6}$$

Proof. There exists a function $u_\varkappa \in \mathscr{K} \cap C^\infty(\bar{\Omega})$ such that

$$\|u - u_\varkappa\|_1 < \tfrac{1}{2}\varepsilon .$$

Let $u_{\varkappa I}$ be the Lagrange linear interpolate of $u_\varkappa$ over $\mathscr{T}_h$, consequently, $u_{\varkappa I} \in \mathscr{K}_h$. For sufficiently small h it holds

$$\|u_{\varkappa I} - u\|_1 \leqq \|u_{\varkappa I} - u_\varkappa\|_1 + \|u_\varkappa - u\|_1 < \varepsilon$$

and (3.2) is satisfied by $v_h \equiv u_{\varkappa I}$. Then (3.6) follows from Theorem 3.1.

4. FINITE ELEMENT APPROXIMATIONS TO THE DUAL PROBLEM

Instead of the dual problem (1.12) we introduce an equivalent problem. To this end, we find a vector $\bar{\lambda} \in Q$ such that

$$\operatorname{div} \bar{\lambda} + f = 0 \quad \text{in} \quad \Omega .$$

We show that a vector $\mathbf{z}^0 \in Q$ exists such that

$$\operatorname{div} \mathbf{z}^0 = 0 \quad \text{in} \quad \Omega , \tag{4.1}$$

$$\mathbf{z}^0 . \nu|_\Gamma = -\bar{\lambda} . \nu - g_0 ,$$

where

$$g_0 = (f, 1)_0/\text{mes}\, \Gamma = \text{const} < 0 .$$

Then the sum $\lambda^f = \bar{\lambda} + \mathbf{z}^0 \in Q$ satisfies the conditions

$$\operatorname{div} \lambda^f + f = 0 \quad \text{in} \quad \Omega , \tag{4.2}$$

$$\lambda^f . \nu|_\Gamma = -g_0 ,$$

hence $\lambda^f \in \mathscr{U}$.

The vector-function $\mathbf{z}^0$ can be defined e.g. as $\mathbf{z}^0 = \text{grad}\, w$, where

$$\Delta w = 0 \text{ in } \Omega , \quad \left.\frac{\partial w}{\partial \nu}\right|_\Gamma = -\bar{\lambda} . \nu - g_0 .$$

Such function w exists, because we have

$$\langle \bar{\lambda} . \nu + g_0, 1 \rangle = \langle \bar{\lambda} . \nu, 1 \rangle + (f, 1)_0 = (f + \operatorname{div} \bar{\lambda}, 1)_0 = 0 .$$

Since we need an explicit λ^f in what follows (see Remark 4 below), a $\mathbf{z}^0$ has to be constructed. In case that $\bar{\lambda} . \nu$ is piecewise linear on Γ, we are able to find a $\mathbf{z}^0 \in \mathcal{N}_h(\Omega)$ (cf. [5] for the definition of $\mathcal{N}_h(\Omega)$) such that (4.1) is satisfied.

If $\bar{\lambda} . \nu$ is not piecewise linear on Γ, we can also use the following approach. Let us find a function $\omega \in H^2(\Omega)$ satisfying the relation

$$\omega(s) = -\int_{s_0}^{s} (\bar{\lambda} . \nu + g_0)\, \mathrm{d}t \quad \forall s \in \Gamma .$$

Then the vector $\mathbf{z}^0 = \{-\partial\omega/\partial x_2, \partial\omega/\partial x_1\}$ satisfies the boundary condition

$$\frac{\partial\omega}{\partial s} = -\frac{\partial\omega}{\partial x_2} \nu_1 + \frac{\partial\omega}{\partial x_1} \nu_2 = \mathbf{z}^0 . \nu = -\bar{\lambda} . \nu - g_0 .$$

(The function ω can be sought by a finite element method, using e.g. quintic polynomials over a suitable triangulation with zero nodal parameters inside Ω.)

It is readily seen that the problem to find a $\boldsymbol{q}^0 \in \mathcal{U}_0 = \{\boldsymbol{q} \mid \boldsymbol{q} \in Q, \operatorname{div} \boldsymbol{q} = 0 \text{ in } \Omega, (\boldsymbol{q} + \lambda^f) . \nu|_\Gamma \geqq 0\}$ such that

$$J(\boldsymbol{q}^0) \leqq J(\boldsymbol{q}) \quad \forall \boldsymbol{q} \in \mathcal{U}_0 , \tag{4.3}$$

where

$$J(\boldsymbol{q}) = \tfrac{1}{2}\|\boldsymbol{q}\|^2 + ((\lambda^f, \boldsymbol{q})) \quad \text{and} \quad ((\boldsymbol{q}, \boldsymbol{p})) = \sum_{i=1}^{2} (q_i, p_i)_0, \; \|\boldsymbol{q}\|^2 = ((\boldsymbol{q}, \boldsymbol{q})) ,$$

is *equivalent* with the dual problem (1.12).

The solutions satisfy the relation

$$\lambda^0 = \lambda^f + \boldsymbol{q}^0 .$$

Let us introduce the convex set

$$\mathcal{U}_0^h = \{\boldsymbol{q} \mid \boldsymbol{q} \in \mathcal{N}_h(\Omega), \boldsymbol{q} . \nu|_\Gamma \geqq g_0\} = \mathcal{U}_0 \cap \mathcal{N}_h(\Omega) .$$

We say that a vector $\lambda^f + \boldsymbol{q}^h$, $\boldsymbol{q}^h \in \mathcal{U}_0^h$ is a *finite element approximation to the dual problem*, if

$$J(\boldsymbol{q}^h) \leqq J(\boldsymbol{q}) \quad \forall \boldsymbol{q} \in \mathcal{U}_0^h . \tag{4.4}$$

The problem (4.4) has a unique solution. In fact, $\mathcal{U}_0^h$ is non-empty, containing the zero vector. $J(\boldsymbol{q})$ is continuously differentiable and strictly convex in $[L_2(\Omega)]^2$, $\mathcal{U}_0^h$ closed and convex. Hence the existence and uniqueness of q^h follows.

Lemma 4.1. *Suppose there exists a* $\mathbf{W}^h \in \mathcal{U}_0^h$ *such that* $2\boldsymbol{q}^0 - \mathbf{W}^h \in \mathcal{U}_0$. *Then it holds*

$$\|\boldsymbol{q}^0 - \boldsymbol{q}^h\| \leqq \|\boldsymbol{q}^0 - \mathbf{W}^h\| . \tag{4.5}$$

(For the proof – see Lemma 2.1 of [1], where $B = [L_2(\Omega)]^2$, $\mathscr{J} = J$, $M = \mathscr{U}_0$, $M_h = \mathscr{U}_0^h$, $\alpha_0 = c = 1$.)

Lemma 4.2. Let $\boldsymbol{q}^0 \in [H^2(\Omega)]^2$, $\boldsymbol{q}^0 . \nu \in H^2(\Gamma_m)$ for any side Γ_m, $m = 1, \ldots, G$ of the polygonal boundary Γ. Then for sufficiently small h there exists a piecewise linear function ψ_h on Γ, with the nodes determined by the vertices of $\mathscr{T}_h$ and such that

$$\int_\Gamma \psi_h \, \mathrm{d}s = \int_\Gamma \boldsymbol{q}^0 . \nu \, \mathrm{d}s = 0 , \tag{4.6}$$

$$g_0 \leqq \psi_h \leqq 2\boldsymbol{q}^0 . \nu - g_0 \quad \text{on} \quad \Gamma , \tag{4.7}$$

$$\|\psi_h - (r_h \boldsymbol{q}^0) . \nu\|_{L_2(\Gamma)} \leqq C h^2 \sum_{m=1}^{G} |\boldsymbol{q}^0 . \nu|_{2,\Gamma_m} , \tag{4.8}$$

where r_h is the projection mapping $\boldsymbol{q}^0$ into $\mathscr{N}_h(\Omega)$ (cf. [5] or [1] – Section 4) and $|\cdot|_{2,\Gamma_m}$ the seminorm generated by the second derivatives with respect to the arc-parameter.

Remark 4.1. In comparison with [1], here the one-sided approximations of the flux $\boldsymbol{q}^0 . \nu$ cannot be used. In fact, setting

$$g_0 \leqq \psi_h \leqq \boldsymbol{q}^0 . \nu \quad \text{on} \quad \Gamma ,$$

and (4.6), we obtain

$$0 \leqq \int_\Gamma (\boldsymbol{q}^0 . \nu - \psi_h) \, \mathrm{d}s = 0 \Rightarrow \psi_h = \boldsymbol{q}^0 . \nu$$

which is impossible, in general, as $\boldsymbol{q}^0 . \nu$ need not be piecewise linear on Γ.

Proof of lemma 4.2. For brevity, let us denote $\boldsymbol{q}^0 . \nu = t$. According to the definition of r_h, the linear function $(r_h \boldsymbol{q}^0) . \nu$ is determined by the $L_2(S_k)$-projection of t into $P_1(S_k)$ on every side $S_k \subset \Gamma$ of the triangulation $\mathscr{T}_h$. Denote also $(r_h \boldsymbol{q}^0) . \nu = t_h$.

It is easy to see that the solution u of the primary problem has the following property, provided $(f, 1)_0 < 0$: $\partial u / \partial \nu > 0$ holds on $E \subset \Gamma$, mes $E > 0$. From Theorem 1.1 we conclude that $\lambda^0 . \nu = \partial u / \partial \nu > 0$ on E,

$$t = \boldsymbol{q}^0 . \nu = (\lambda^0 - \lambda^f) . \nu = \lambda^0 . \nu + g_0 > g_0 \quad \text{on} \quad E ,$$

$$t = g_0 \quad \text{on} \quad \Gamma \dot{-} E .$$

From the assumption $t \in H^2(\Gamma_m)$ it follows $t \in C^1(\bar{\Gamma}_m)$ for all m, consequently

$$\operatorname{supp}(t - g_0) = \bigcup_{m=1}^{G} \bigcup_j I_j^{(m)} ,$$

where $I_j^{(m)} \subset \bar{\Gamma}_m$ are closed intervals of positive length.

Consider an arbitrary interval $I_j^{(m)} \equiv \langle \sigma, \bar{\sigma} \rangle$ and let $s_0 \leqq \sigma < s_1 < \bar{\sigma}$, where $\langle s_{k-1}, s_k \rangle$ corresponds with a side $S_k \in \mathscr{T}_h$, $(k = 1, 2, \ldots)$. Then we set $\psi_h = g_0$ on $\langle s_0, s_2 \rangle$. (In case that $\lim_{j \to \infty} (\text{mes}\, I_j^{(m)}) = 0$, $I_j^{(m)} = \langle \sigma_j, \bar{\sigma}_j \rangle$, $\sigma_j \to \sigma$, $\bar{\sigma}_j \to \sigma$, $\lim_{s \to \sigma+} (t - g_0)(s) = 0$, we also set $\psi_h = g_0$ on a suitable interval $\langle s_0, s_k \rangle$, where $t(s_k) > g_0$).

Let $t - g_0 > 0$ at all vertices $Q_k \in \mathscr{T}_h$ with parameters $s_1 < s_2 < \ldots < s_{n-1} \angle \bar{\sigma}$ and let $\bar{\sigma} \leqq \sigma_n$. We set $\psi_h = g_0$ on $\langle s_{n-2}, s_n \rangle$ and $\psi_h = t_h + a_j$ in $\langle s_{k-1}, s_k \rangle$ for $k = 3, 4, \ldots, n - 2$, where

$$a_j = (s_{n-2} - s_2)^{-1} \left\{ \int_\sigma^{s_2} (t - g_0) \, \mathrm{d}s + \int_{s_{n-2}}^{\bar{\sigma}} (t - g_0) \, \mathrm{d}s \right\} \tag{4.9}$$

(provided $s_{n-2} > s_2$). There exists a point $\vartheta \in \langle \sigma, s_2 \rangle$ such that

$$\int_\sigma^{s_2} (t - g_0) \, \mathrm{d}s = (t - g_0)(\vartheta)(s_2 - \sigma) \tag{4.10}$$

and it holds

$$(t - g_0)(\xi) = \int_\sigma^\xi \frac{\mathrm{d}^2 t}{\mathrm{d}s^2}(s)(\xi - s) \, \mathrm{d}s \leqq (2h)^{3/2} \|t''\|_{L_2(\Gamma_m)} \quad \forall \xi \in \langle \sigma, s_2 \rangle . \tag{4.11}$$

From there we obtain an upper bound for the first integral in (4.10). The second integral can be estimated in a similar way. Consequently, we have

$$a_j \leqq 2^{5/2} (s_{n-2} - s_2)^{-1} h^{5/2} \|t''\|_{L_2(\Gamma_m)} .$$

Denoting $l_j = \bar{\sigma} - \sigma$ the length of $I_j^{(m)}$, we obtain for sufficiently small h

$$(s_{n-2} - s_2)^{-1} \leqq (l_j - 4h)^{-1} \leqq 2/l_j .$$

Without any loss of generality, a finite number of intervals $I_j^{(m)}$ can be considered and therefore

$$l_j \geqq \min l_j = c > 0 .$$

(In case that $l_j \to 0$ for $j \to \infty$, we substitute the interval $I_j^{(m)}$ by a suitable union $\bigcup_{j=k}^{\infty} I_j^{(m)}$). Thus we obtain

$$a_j \leqq 2^{7/2} c^{-1} h^{5/2} \|t''\|_{L_2(\Gamma_m)} , \tag{4.12}$$

where c does not depend on h.

Let us consider the interval $\langle s_0, s_2 \rangle = S_1 \cup S_2$. We have

$$\|\psi_h - t_h\| \leqq \|g_0 - t\| + \|t - t_h\| ,$$

with $L_2(S_i)$-norms, $i = 1, 2$.

Making use of (4.11), we deduce

$$\|t - g_0\|_{L_2(S_i)} \leqq Ch^2\|t''\|_{L_2(\Gamma_m)}$$

and a similar estimate is true for $\|t - t_h\|_{L_2(S_i)}$. Consequently,

$$\|\psi_h - t_h\|_{L_2(s_0,s_2)} \leqq Ch^2\|t''\|_{L_2(\Gamma_m)} \tag{4.13}$$

holds and an analogous estimate is true for the interval (s_{n-2}, s_n). Altogether, from (4.12) and (4.13) it follows that

$$\|\psi_h - t_h\|^2_{L_2(s_0,s_n)} = \|\psi_h - t_h\|^2_{L_2(s_0,s_2)} + \|\psi_h - t_h\|^2_{L_2(s_{n-2},s_n)} + \int_{s_2}^{s_{n-2}} a_j^2 \, ds \leqq$$

$$\leqq 2Ch^4\|t''\|^2_{L_2(\Gamma_m)} + C_1 l_j \, . \, h^5\|t''\|^2_{L_2(\Gamma_m)} \leqq C_2h^4\|t''\|^2_{L_2(\Gamma_m)} \, .$$

Moreover, we set $\psi_h = g_0$ on $\Gamma_m \dot{-} \bigcup_j I_j^{(m)}$. By virtue of the finite number of intervals considered above, we obtain a similar estimate for $\|\psi_h - t_h\|^2_{L_2(\Gamma_m)}$ and (4.8) follows.

From (4.9) and the well-known relation

$$\int_{S_k} (t_h - t) \, ds = 0 \quad \forall S_k \subset \Gamma \, ,$$

we obtain

$$\int_\sigma^{\bar{\sigma}} (\psi_h - t) \, ds = \int_\sigma^{s_2} (g_0 - t) \, ds + \int_{s_{n-2}}^{\bar{\sigma}} (g_0 - t) \, ds + \int_{s_2}^{s_{n-2}} (\psi_h - t_h) \, ds =$$

$$= \int_\sigma^{s_2} (g_0 - t) \, ds + \int_{s_{n-2}}^{\bar{\sigma}} (g_0 - t) \, ds + a_j(s_{n-2} - s_2) = 0 \, .$$

Hence the condition (4.6) is satisfied. The inequalities (4.7) are also satisfied, if h is sufficiently small.

Theorem 4.1. Let Ω be simply connected, (2.1) hold and the assumptions of Lemma 4.2 be satisfied. Denote $\lambda^h = \lambda^f + \boldsymbol{q}^h$, $\lambda^0 = \lambda^f + \boldsymbol{q}^0$, where λ^f satisfies (4.2), $\boldsymbol{q}^h$ and $\boldsymbol{q}^0$ are solutions of the problems (4.4) and (4.3), respectively. Then for α-β-regular triangulations it holds

$$\|\lambda^h - \lambda^0\| \leqq Ch^{3/2}\{|\boldsymbol{q}^0|_{2,\Omega} + \sum_{m=1}^{G} |\boldsymbol{q}^0 \, . \, \nu|_{2,\Gamma_m}\} \, , \tag{4.14}$$

where $|\boldsymbol{q}^0|_{2,\Omega}$ is the seminorm generated by second derivatives.

Proof. Let ψ_h be the approximation of the flux from Lemma 4.2. We set

$$\varphi = (r_h\boldsymbol{q}^0) \, . \, \nu - \psi_h \equiv t_h - \psi_h \, .$$

There exists a function $w^h \in \mathcal{N}_h(\Omega)$ such that

$$w^h . \nu = \varphi \quad \text{on} \quad \Gamma ,$$

(4.15)
$$\|w^h\| \leqq C h^{-1/2} \|\varphi\|_{L_2(\Gamma)}$$

(see [1] – Lemma 5.3, where $J = 1$), because we have

$$\int_\Gamma (t_h - \varphi_h) \, \mathrm{d}s = \int_\Gamma (t - \psi_h) \, \mathrm{d}s = 0$$

by virtue of (4.6).

The the function $W_h = r_h q^0 - w^h$ satisfies the conditions of Lemma 4.1. In fact, $W_h \in \mathcal{N}_h(\Omega)$,

$$W_h . \nu = t_h - \varphi = \psi_h \geqq g_0 \quad \text{on} \quad \Gamma ,$$

consequently, $W_h \in \mathcal{U}_0^h$. From (4.7) it follows

$$W_h . \nu \leqq 2q^0 . \nu - g_0 \Rightarrow (2q^0 - W_h) . \nu - g_0 \geqq 0 ,$$
$$2q^0 - W_h \in \mathcal{U}_0 .$$

Making use of the estimate (cf. [5] – Th. 3.1)

$$\|q - r_h q\| \leqq C h^2 |q|_{2,\Omega} \quad \forall q \in [H^2(\Omega)]^2$$

and of (4.15), (4.8), we obtain

$$\|q^0 - W_h\| \leqq \|q^0 - r_h q^0\| + \|r_h q - W_h\| \leqq C h^2 |q^0|_{2,\Omega} + \|w_h\| \leqq$$
$$\leqq C h^2 |q^0|_{2,\Omega} + C_1 h^{3/2} \sum_{m=1}^{G} |q^0 . \nu|_{2,\Gamma_m} .$$

Then the estimate (4.14) follows from Lemma 4.1.

5. A POSTERIORI ERROR ESTIMATES AND TWO-SIDED BOUNDS OF ENERGY

The dual analysis enables us to find a posteriori error estimates for the finite element approximations.

From (1.4) we obtain for any $v \in \mathcal{K}$

(5.1)
$$2[\mathcal{L}(v) - \mathcal{L}(u)] = |v|_1^2 - |u|_1^2 - 2(f, v - u)_0 \geqq$$
$$\geqq |v|_1^2 - |u|_1^2 - 2(\text{grad}\, u, \text{grad}\, (v - u)) = |v - u|_1^2 .$$

By virtue of (1.15) we may write

(5.2)
$$-\mathcal{L}(u) = \mathcal{S}(\lambda^0) \leqq \mathcal{S}(\lambda) \quad \forall \lambda \in \mathcal{U} .$$

Theorem 5.1. *Let* $\tilde{u}_h \in \mathscr{K}_h$ *be any approximation to the primary problem and* $\tilde{\lambda}^h = \lambda^f + \tilde{\mathbf{q}}^h$, *where* $\tilde{\mathbf{q}}^h \in \mathscr{U}_0^h$, *any approximation to the dual problem. Then it holds*

$$(5.3)\qquad |\tilde{u}_h - u|_1^2 \leqq \sum_{i=1}^{2} \left\| \tilde{\lambda}_i^h - \frac{\partial \tilde{u}_h}{\partial x_i} \right\|_0^2 + 2 \int_\Gamma \tilde{\lambda}^h . \nu \tilde{u}_h \, ds \equiv E(\tilde{u}_h, \tilde{\lambda}^h),$$

$$\sum_{i=1}^{2} \left\| \tilde{\lambda}_i^h - \frac{\partial u}{\partial x_i} \right\|_0^2 \leqq E(\tilde{u}_h, \tilde{\lambda}^h).$$

Proof. From (5.1) and (5.2) it follows

$$|\tilde{u}_h - u|_1^2 \leqq 2\,\mathscr{L}(\tilde{u}_h) + 2\,\mathscr{S}(\tilde{\lambda}^h) = |\tilde{u}_h|_1^2 - 2(f, u_h)_0 + \|\tilde{\lambda}^h\|^2 =$$
$$= \|\tilde{\lambda}^h - \operatorname{grad} \tilde{u}_h\|^2 + 2((\tilde{\lambda}^h, \operatorname{grad} \tilde{u}_h)) - 2(f, \tilde{u}_h)_0 .$$

On the other hand, we have

$$((\tilde{\lambda}^h, \operatorname{grad} \tilde{u}_h)) - (f, \tilde{u}_h)_0 = -(\operatorname{div} \tilde{\lambda}^h + f, \tilde{u}_h)_0 + \int_\Gamma \tilde{\lambda}^h . \nu u_h \, ds .$$

Using (4.2), we obtain

$$\operatorname{div} \tilde{\lambda}^h + f = \operatorname{div} \lambda^f + f = 0$$

and we are led to (5.3).

The solution λ^0 of (1.12) satisfies the inequality

$$((\lambda^0, \lambda - \lambda^0)) \geqq 0 \quad \forall \lambda \in \mathscr{U} .$$

Consequently, for any $\lambda \in \mathscr{U}$ we may write

$$2[\mathscr{S}(\lambda) - \mathscr{S}(\lambda^0)] = \|\lambda\|^2 - \|\lambda^0\|^2 \geqq \|\lambda\|^2 - ((\lambda^0, \lambda)) =$$
$$= ((\lambda, \lambda - \lambda^0)) - ((\lambda^0, \lambda - \lambda^0)) + ((\lambda^0, \lambda - \lambda^0)) \geqq \|\lambda - \lambda^0\|^2 .$$

Inserting $\lambda = \tilde{\lambda}^h$ and using (1.14), (1.15), we obtain

$$\|\tilde{\lambda}^h - \operatorname{grad} u\|^2 \leqq 2\,\mathscr{S}(\tilde{\lambda}^h) + 2\,\mathscr{L}(u) \leqq 2\,\mathscr{S}(\tilde{\lambda}^h) + 2\,\mathscr{L}(\tilde{u}^h) = E(\tilde{u}_h, \tilde{\lambda}^h).$$

Remark 5.1. The upper bound $E(\tilde{u}_h, \tilde{\lambda}^h)$ consists of non-negative terms. It is not the case for the bound $2\,\mathscr{L}(\tilde{u}_h) + 2\,\mathscr{S}(\tilde{\lambda}^h)$. In fact, $\mathscr{L}(\tilde{u}_h) \to \mathscr{L}(u) = -\frac{1}{2}|u|_1^2$ (cf. (3.4), (3.5) and (1.6)), and consequently, $\mathscr{L}(\tilde{u}_h)$ is negative, in practice.

Theorem 5.2. *Under the assumptions of Theorem* 5.1 *the following two-sided energy estimates hold*:

$$-2\,\mathscr{L}(\tilde{u}_h) \leqq |u|_1^2 \leqq 2\,\mathscr{S}(\tilde{\lambda}^h),$$
$$-2\,\mathscr{L}(\tilde{u}_h) \leqq (f, u)_0 \leqq 2\,\mathscr{S}(\tilde{\lambda}^h).$$

Proof. By virtue of (1.6) we may write

$$2\,\mathscr{L}(u) = |u|_1^2 - 2(f, u)_0 = -|u|_1^2 \leqq 2\,\mathscr{L}(\tilde{u}_h)\,.$$

Using (1.15), we obtain

$$|u|_1^2 = -2\,\mathscr{L}(u) = 2\,\mathscr{S}(\lambda^0) \leqq 2\,\mathscr{S}(\bar{\lambda}^h) \quad \forall \bar{\lambda}^h \in \mathscr{U}\,.$$

References

[1] *Hlaváček, I.:* Dual finite element analysis for unilateral boundary value problems. Apl. mat. 22 (1977), 14—51.

[2] *Hlaváček, I.:* Dual finite element analysis for elliptic problems with obstacles on the boundary I. Apl. mat. 22 (1977), 244—255

[3] *Mosco, U.* and *Strang, G.:* One-sided approximations and variational inequalities. Bull. Amer. Math. Soc. 80 (1974), 308—312.

[4] *Duvaut, G.* and *Lions, J. L.:* Les inéquations en mécanique et en physique. Dunod, Paris 1972.

[5] *Haslinger, J.* and *Hlaváček, I.:* Convergence of a finite element method based on the dual variational formulation. Apl. mat. 21 (1976), 43—65.

[6] *Céa, J.:* Optimisation, théorie et algorithmes. Dunod, Paris 1971.

[7] *Falk, R. S.:* Error estimates for the approximation of a class of variational inequalities. Math. Comp. 28 (1974), 963—971.

[8] *Fichera, G.:* Boundary value problems of elasticity with unilateral constraints. Encycl. of Physics (ed. S. Flügge), vol. VI a/2. Springer, Berlin 1972.

[9] *Nečas, J.:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.

Souhrn

DUÁLNÍ ANALÝZA SEMI-KOERCIVNÍCH ÚLOH S JEDNOSTRANNÝMI OKRAJOVÝMI PODMÍNKAMI METODOU KONEČNÝCH PRVKŮ

IVAN HLAVÁČEK

Duální analýza koercivních jednostranných úloh byla zavedena autorem v článcích [1] a [2]. Tam byly odvozeny některé a priorní odhady chyb za předpokladu regularity řešení. A posteriorní odhady chyb plynou pak z duálního přístupu. V této práci je duální analýza rozšířena na semi-koercivní úlohy s homogenními jednostrannými podmínkami na hranici oblasti. Pomocí metody Falkovy [7] a Moscovy-Strangovy [3] odvozují se analogické apriorní odhady jako v [1]. Dále je dokázána konvergence aproximací primární úlohy bez předpokladu regularity řešení.

Author's address: Ing. *Ivan Hlaváček*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.