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ORTHOEXPONENTIAL POLYNOMIALS AND THE LEGENDRE POLYNOMIALS

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Orthogonal exponential polynomials or *orthoexponential polynomials* have interesting applications in Automatic Control – Laning-Battin [8], Electrical Circuits Theory – Čížek [2], Tuttle [10], Hydrometeorology – Dmitriyev [4], and as a means for the inversion of Laplace transforms – Jaroch [5]. A very simple connection will be shown to exist between the orthoexponential and the Legendre polynomials.

In this paper, the letters k, m, n denote nonnegative integers, t and x are real variables, δ_{mn} is the Kronecker delta function and, in Definition II only, $[n/2]$ denotes the integer part of $n/2$. As the notation and standardization of both the orthoexponential and the Legendre polynomials vary, definitions of the Legendre, Jacobi, and orthoexponential polynomials as used in this paper are re-stated here.

Definition I [6]. *The functions* $\varphi_n(t) = \sum_{k=1}^{n} b_{nk} e^{-kt}$, $n = 1, 2, 3, \ldots$, *are called orthoexponential polynomials if the coefficients* b_{nk} *are so chosen that for all* m, n *the following conditions are satisfied*: $\varphi_n(0) = 1$ *and* $\int_0^\infty \varphi_m(t)\, \varphi_n(t)\, \mathrm{d}t = \delta_{mn} \|\varphi_n\|^2$.

The coefficients of orthoexponential polynomials are $b_{nk} = (-1)^{n+k} \binom{n}{k} \binom{n+k-1}{k-1}$; $\varphi_1(t) = e^{-t}$. For all n we have $\varphi_n(+\infty) = 0$. Orthoexponential polynomials are bounded and $|\varphi_n(t)| \leqq 1$ for all nonnegative t. Their norm is $\|\varphi_n\| = (2n)^{-1/2}$ and orthonormal exponential polynomials are therefore $t \mapsto (2n)^{1/2}\, \varphi_n(t)$.

Definition II [9]. *The polynomials* $P_n(x) = \sum_{k=0}^{[n/2]} a_{nk} x^{n-2k}$, $n = 0, 1, 2, \ldots$, *are called Legendre polynomials if the coefficients* a_{nk} *are so chosen that for all* m, n *the following conditions are satisfied*: $P_n(1) = 1$ *and* $\int_{-1}^{+1} P_m(x)\, P_n(x)\, \mathrm{d}x = \delta_{mn} \|P_n\|^2$.

The coefficients of the Legendre polynomials are $a_{nk} = (-1)^k\, 2^{-n} \binom{n}{k} \binom{2n-2k}{n}$; $P_0(x) \equiv 1$. Their norm is $\|P_n\| = (n + \frac{1}{2})^{-1/2}$, and $|P_n(x)| \leqq 1$ if only $-1 \leqq x \leqq 1$.

The notation $P_n^{(\alpha,\beta)}(x)$ is used for the Jacobi polynomials, orthogonal over $(-1, 1)$ with respect to the weight function $(1 - x)^\alpha (1 + x)^\beta$, $\alpha > -1$, $\beta > -1$, and standardized with $P_n^{(\alpha,\beta)}(1) = \binom{n + \alpha}{n}$. Legendre polynomials are a special case of the Jacobi polynomials for $\alpha = \beta = 0$. Many properties of the Jacobi polynomials originate from their close connection with the hypergeometric function. The following relation will be used in this paper:

$$(1) \qquad (1 + x)\, P_{n-1}^{(0,1)}(x) = P_n^{(0,0)}(x) + P_{n-1}^{(0,0)}(x)\,, \quad n \geqq 1\,,$$

where $P_n^{(0,0)}(x) \equiv P_n(x)$ are the Legendre polynomials [1], [9].

It is known [6] that orthoexponential polynomials can be expressed in terms of the Jacobi polynomials $P_n^{(0,1)}(x)$. However, no explicit mention has been found so far of a simple relation connecting the orthoexponential and the Legendre polynomials as expressed in the following two theorems.

Theorem 1. *Let $\varphi_n(t)$ be the orthoexponential polynomials and $P_n(x)$ the Legendre polynomials in accordance with Definitions I and II; $n = 1, 2, 3, \ldots$. Then, for arbitrary t,*

$$(2) \qquad \varphi_n(t) = \tfrac{1}{2}[P_n(2e^{-t} - 1) + P_{n-1}(2e^{-t} - 1)]\,.$$

Proof. In Eq. (2), orthoexponential polynomials are expressed, in accordance with Definitions I and II, as linear combinations of exponential functions $e^{-t}, e^{-2t}, \ldots$ $\ldots, e^{-nt}$. The standardization $\varphi_n(0) = 1$, $\varphi_n(+\infty) = 0$, is satisfied at the same time. It remains to prove that $\int_0^\infty \varphi_m(t)\, \varphi_n(t)\, \mathrm{d}t = \delta_{mn}(2n)^{-1}$. Substituting $x = 2e^{-t} - 1$ in Eq. (1) we have

$$2e^{-t} P_{n-1}^{(0,1)}(2e^{-t} - 1) = P_n(2e^{-t} - 1) + P_{n-1}(2e^{-t} - 1)$$

and Eq. (2) is therefore equivalent to

$$(3) \qquad \varphi_n(t) = e^{-t} P_{n-1}^{(0,1)}(2e^{-t} - 1)\,.$$

The norm of the Jacobi polynomials is known [1] [9] and for the case under consideration we have $\|P_n^{(0,1)}\| = 2^{1/2}(n + 1)^{-1/2}$. As a consequence,

$$\int_0^\infty \varphi_m(t)\, \varphi_n(t)\, \mathrm{d}t = \int_0^\infty e^{-2t} P_{m-1}^{(0,1)}(2e^{-t} - 1)\, P_{n-1}^{(0,1)}(2e^{-t} - 1)\, \mathrm{d}t$$

and, if the substitution $x = 2e^{-t} - 1$ is used,

$$\int_0^\infty \varphi_m(t)\, \varphi_n(t)\, \mathrm{d}t = \frac{1}{4} \int_{-1}^{+1} (1 + x)\, P_{m-1}^{(0,1)}(x)\, P_{n-1}^{(0,1)}(x)\, \mathrm{d}x = \delta_{mn}(2n)^{-1}\,.$$

Q.e.d.

Remark. Jacobi polynomials are defined by Courant and Hilbert [3] [6] as polynomials orthogonal over $(0, 1)$ with respect to the weight function $x^{q-1}(1-x)^{p-q}$, $q > 0$, $p - q > -1$, and denoted by $G_n(p, q, x)$. The standardization is $G_n(p, q, 0) = 1$ for all n. In this case Eq. (3) becomes

$$\varphi_n(t) = (-1)^{n-1}\, n e^{-t} G_{n-1}(2, 2, e^{-t}) \tag{4}$$

and Eq. (2) assumes the form

$$\varphi_n(t) = (-1)^n\, 2^{-1}[G_n(1, 1, e^{-t}) - G_{n-1}(1, 1, e^{-t})]\,. \tag{5}$$

Example 1. The value of $\varphi_{10}(1)$ is to be computed from Eq. (2). For $t = 1$ we have $2e^{-t} - 1 = -0{\cdot}264241$ and thus $\varphi_{10}(1) = \frac{1}{2}[P_{10}(-0{\cdot}264241) + P_9(-0{\cdot}264241)] = \frac{1}{2}(0{\cdot}237018 - 0{\cdot}148194)$. The result is $\varphi_{10}(1) = 0{\cdot}044412$. A calculation of values of the orthoexponential polynomials based on Definition I involves, for large n, computations with prohibitively large numbers. In this example, if Definition I is used for the computation of $\varphi_{10}(1)$ then the largest coefficient involved is $b_{10,7} = -960960$.

Theorem 2. *Let $\varphi_n(t)$ be the orthoexponential polynomials and $P_n(x)$ the Legendre polynomials in accordance with Definitions I and II; $n = 1, 2, 3, \ldots$. Then, for arbitrary t,*

$$P_n(2e^{-t} - 1) = 2\,\varphi_n(t) - 2\,\varphi_{n-1}(t) + \ldots + (-1)^{n-1}\, 2\,\varphi_1(t) + (-1)^n\,. \tag{6}$$

Proof. For $n = 1$ we have $\varphi_1(t) = \frac{1}{2}[P_1(2e^{-t} - 1) + 1]$ as a consequence of Theorem 1 and thus $P_1(2e^{-t} - 1) = 2\,\varphi_1(t) - 1$. For $n = 2$ we find $\varphi_2(t) = \frac{1}{2}[P_2(2e^{-t} - 1) + P_1(2e^{-t} - 1)]$ or $\varphi_2(t) = \frac{1}{2}[P_2(2e^{-t} - 1) + 2\,\varphi_1(t) - 1]$ and that too agrees with Eq. (6). The proof proceeds by induction. Let us assume that Eq. (6) is valid for a certain integer $n > 1$. It follows from Theorem 1 that $\varphi_{n+1}(t) = \frac{1}{2}[P_{n+1}(2e^{-t} - 1) + P_n(2e^{-t} - 1)]$ or $P_{n+1}(2e^{-t} - 1) = 2\,\varphi_{n+1}(t) - P_n(2e^{-t} - 1)$. Substituting for $P_n(2e^{-t} - 1)$ from Eq. (6) we have $P_{n+1}(2e^{-t} - 1) = 2\,\varphi_{n+1}(t) - 2\,\varphi_n(t) + \ldots + (-1)^n\, 2\,\varphi_1(t) + (-1)^{n+1}$. Q.e.d.

Remark. For the Jacobi polynomials in the Courant and Hilbert notation [3] an analogous relation follows from Eq. (5):

$$G_n(1, 1, e^{-t}) = 1 - 2\,\varphi_1(t) + 2\,\varphi_2(t) - 2\,\varphi_3(t) + \ldots + (-1)^n\, 2\,\varphi_n(t)\,. \tag{7}$$

Example 2. The series $1 - 2\,\varphi_1(t) + 2\,\varphi_2(t) - 2\,\varphi_3(t) + \ldots + (-1)^n\, 2\,\varphi_n(t) + \ldots = 0$ is convergent for any positive t. The n-th partial sum of this series is $(-1)^n \,.\, P_n(2e^{-t} - 1)$ as a consequence of Theorem 2. Laplace's theorem on the asymptotics of the Legendre polynomials [9] states that $P_n(x) = O(n^{-1/2})$ for $n \to \infty$ if only $-1 < x < 1$. Thus, the n-th partial sum of the series under consideration is $(-1)^n\,.$

. $P_n(2e^{-t} - 1) = o(1)$ and the series therefore converges. Accordingly, the series $2\sum_{n=1}^{\infty}(-1)^{n-1}\varphi_n(t) = 1$ converges for any positive t; for $t = 0$ the series is divergent.

CONCLUSION

The theorems proved in this paper are useful if a formulation is sought, for orthoexponential polynomials, of some of the many known results in the theory of classical orthogonal polynomials. Furthermore, Theorem 1 may be convenient for the computation of values of orthoexponential polynomials if the values of the Legendre polynomials are available. In addition to the recurrence formula

$$(8)\qquad (n+1)(2n-1)\varphi_{n+1} = \\ = [(4n^2-1)e^{-t} - 2n^2]\,2\varphi_n - (n-1)(2n+1)\varphi_{n-1},$$

$n = 1, 2, 3, \ldots$; $\varphi_1 = e^{-t}$, $\varphi_0 \equiv 0$ (see Jaroch-Novotný [7]) we have here another relation, namely $\varphi_n(t) = \frac{1}{2}[P_n(2e^{-t}-1) + P_{n-1}(2e^{-t}-1)]$, which is useful when an effective algorithm is required for the computation of values of orthoexponential polynomials.

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Souhrn

O SOUVISLOSTI ORTOEXPONENCIÁLNÍCH MNOHOČLENŮ S LEGENDREOVÝMI MNOHOČLENY

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Vyjádření ortoexponenciálních mnohočlenů $\varphi_n(t) = \sum_{k=1}^{n} b_{nk} e^{-kt}$ podle definice I je pro velká n nevhodné pro numerický výpočet funkčních hodnot, protože k výsledku přicházíme odčítáním velkých čísel. V tomto případě je užitečný rekurentní vzorec (8) nebo v tomto článku dokázaná věta 1, totiž $\varphi_n(t) = \frac{1}{2}[P_n(2e^{-t} - 1) + P_{n-1}(2e^{-t} - 1)]$, kde $P_n(x)$ jsou Legendreovy mnohočleny. Věty, ukazující na souvislost ortoexponenciálních mnohočlenů s Legendreovými mnohočleny jsou užitečné jednak pro numerické výpočty jednak proto, že umožňují jednoduchým způsobem přenést na ortoexponenciální mnohočleny některé výsledky z teorie klasických ortogonálních mnohočlenů. Ortoexponenciální mnohočleny jsme definovali se standardizací $\varphi_n(0) = 1$ pro všechna n, protože při této volbě zvláště vyniknou souvislosti s klasickou teorií ortogonálních mnohočlenů.

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