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ON ASYMPTOTIC STABILITY OF PASSIVE LINEAR ELECTRICAL NETWORKS

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1. INTRODUCTION

Given an oriented graph  $G$  with branches (edges)  $v_1, \dots, v_r$  and nodes (vertices)  $u_1, \dots, u_s$ , let us denote  $\mathbf{v} = (v_1, \dots, v_r)^T$ ,  $\mathbf{u} = (u_1, \dots, u_s)^T$ . Let  $\mathbf{c}$  be a real vector of type  $(r, 1)$ . Then the expression  $K = \mathbf{c}^T \mathbf{v}$  will be called a 1-complex. If  $\tilde{\mathbf{c}} = \tilde{\mathbf{c}}^T \mathbf{v}$  is also a 1-complex,  $\alpha, \tilde{\alpha}$  real numbers, let us define  $\alpha K + \tilde{\alpha} \tilde{K} = (\alpha \mathbf{c} + \tilde{\alpha} \tilde{\mathbf{c}})^T \mathbf{v}$ . We put  $K = \mathbf{c}^T \mathbf{v} = 0$  if and only if  $\mathbf{c} = \mathbf{0}$ . We call the complexes  $K_1, \dots, K_m$  linearly independent, if  $\sum_{i=1}^r \delta_i K_i = 0$  implies that  $\delta_i = 0, i = 1, \dots, r$ . Similarly, the expression  $L = \mathbf{c}^T \mathbf{u}$ , where  $\mathbf{c}$  is a real vector of type  $(s, 1)$ , will be called a 0-complex. The notions of  $\alpha L + \tilde{\alpha} \tilde{L}, L = 0$  and linear independence are defined analogously.

For each branch  $v$  of  $G$  we define  $\partial v = u_2 - u_1$ , where  $u_2(u_1)$  is the terminal (initial) node of the branch  $v$ . For an arbitrary 1-complex  $K = \mathbf{c}^T \mathbf{v}$  we define  $\partial K = \sum_{i=1}^r c_i \partial v_i$ . If  $\partial K = 0$ , then the 1-complex  $K$  will be called a cycle.

**Lemma 1.** Let  $K = \mathbf{c}^T \mathbf{v}$  be a cycle. Then there exist loops  $K_i = \mathbf{d}_i^T \mathbf{v}, i = 1, \dots, m$  such that

1.  $K = \sum_{i=1}^m \alpha_i \mathbf{d}_i^T \mathbf{v}$ ,
2. if we denote  $\mathbf{d}_i = (d_{i1}, \dots, d_{ir})^T, \mathbf{c} = (c_1, \dots, c_r)^T$ , then  $d_{ij} \neq 0 \Rightarrow c_j \neq 0$  for  $i = 1, \dots, m, j = 1, \dots, r$ .

Proof may be found in [2], Theorem 1.2. From Lemma 1 one obtains easily

**Lemma 2.** Let  $\mathbf{B}$  be a real diagonal positive semidefinite matrix of type  $(r, r)$ . Then the condition  $\mathbf{d}^T \mathbf{B} \mathbf{d} > 0$  holds for every loop  $\mathbf{d}^T \mathbf{v}$  if and only if  $\mathbf{c}^T \mathbf{B} \mathbf{c} > 0$  holds for every nonzero cycle  $\mathbf{c}^T \mathbf{v}$ .

The *incidence matrix*  $\mathbf{A} = (a_{ik})$  (of type  $(r, s)$ ) of a graph  $G$  is defined by the conditions

$$\begin{aligned} a_{ik} &= 1, \text{ if } u_k \text{ is the terminal node of the branch } v_i, \\ a_{ik} &= -1, \text{ if } u_k \text{ is the initial node of the branch } v_i, \\ a_{ik} &= 0, \text{ if } u_k \text{ is not incident with the branch } v_i. \end{aligned}$$

**Lemma 3.** *Let  $K = \mathbf{c}^T \mathbf{v}$ ; then  $K$  is a cycle if and only if  $\mathbf{A}^T \mathbf{c} = \mathbf{o}$ .*

Proof is evident.

Let us denote by  $\mathbf{X}$  the matrix of type  $(r, n)$  the columns of which form a complete system of linearly independent solutions of the equation  $\mathbf{A}^T \mathbf{x} = \mathbf{o}$ . Then the following statement is true:

**Lemma 4.** *a) The elements of the vector  $\mathbf{X}^T \mathbf{v}$  form a complete set of linearly independent cycles of the graph  $G$ .*

*b) If  $\mathbf{c}^T \mathbf{v}$  is a cycle, then there exists a real vector  $\mathbf{w}$  such that  $\mathbf{c} = \mathbf{X} \mathbf{w}$ .*

Proof see in [1], Theorem 1.1.

Let  $G$  be an oriented graph,  $\mathbf{R}, \mathbf{L}, \mathbf{S}$  real matrices of type  $(r, r)$ . Then the ordered tetrad  $(G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  will be called a *network*.

Let us denote by  $R$  the field of rational functions of complex variable  $p$  with real coefficients. If  $\mathbf{M}$  is a matrix the elements of which belong to  $R$ , we call it a matrix over  $R$ .

Let a network  $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  be given, let  $\mathbf{e}$  be a vector of type  $(r, 1)$  over  $R$ , let  $\mathbf{i}_0, \mathbf{q}_0$  be constant real vectors of type  $(r, 1)$ . Then a vector  $\mathbf{i}$  of type  $(r, 1)$  over  $R$  is said to be a *solution of the network  $N$  corresponding to the vector  $\mathbf{e}$  and initial vectors  $\mathbf{i}_0, \mathbf{q}_0$* , if the following conditions are satisfied:

$$(K1) \quad \mathbf{A}^T \mathbf{i} = \mathbf{o},$$

$$(K2) \quad \mathbf{c}^T (\mathbf{L}p + \mathbf{R} + \mathbf{S}p^{-1}) = \mathbf{c}^T (\mathbf{e} + \mathbf{L}\mathbf{i}_0 - \mathbf{S}\mathbf{q}_0 p^{-1}) \text{ for every cycle } \mathbf{c}^T \mathbf{v} \text{ of the graph } G.$$

**Theorem 1.** *Let a network  $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  be given, let  $\mathbf{X}$  be a matrix the columns of which form a complete set of linearly independent solutions of the equation  $\mathbf{A}^T \mathbf{x} = \mathbf{o}$ . Then the solution of the network  $N$  corresponding to the vector  $\mathbf{e}$  and initial vectors  $\mathbf{i}_0, \mathbf{q}_0$  (if it exists) is given by*

$$\mathbf{i} = \mathbf{X} [\mathbf{X}^T (\mathbf{L}p + \mathbf{R} + \mathbf{S}p^{-1}) \mathbf{X}]^{-1} \mathbf{X}^T (\mathbf{e} + \mathbf{L}\mathbf{i}_0 - \mathbf{S}\mathbf{q}_0 p^{-1}).$$

Proof follows from [1], Theorem 1.3.

A network  $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  will be called *passive*, if the following conditions are fulfilled:

- a) the matrices  $\mathbf{R}, \mathbf{S}$  are diagonal,
- b) the matrices  $\mathbf{R}, \mathbf{L}, \mathbf{S}$  are positive semidefinite.

**Theorem 2.** Let  $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  be a passive network. If

$$\mathbf{c}^\top(\mathbf{R} + \mathbf{L} + \mathbf{S})\mathbf{c} > 0$$

for every nonzero cycle  $\mathbf{c}^\top \mathbf{v}$  of the graph  $G$ , then for any vectors  $\mathbf{e}, \mathbf{i}_0, \mathbf{q}_0$  there exists a unique solution  $\mathbf{i}$  of  $N$ .

Proof follows from [1], Theorem 5.2.

Let  $\mathbf{Z}$  be a matrix over  $R$ . A complex number  $\alpha$  will be called a pole of  $m$ -th order of the matrix  $\mathbf{Z}$ , if  $\alpha$  is a pole of  $m$ -th order of at least one element of  $\mathbf{Z}$  and a pole of at most  $m$ -th order of each element of  $\mathbf{Z}$ .

Let us denote by  $\mathfrak{G}$  the set of all complex numbers with positive real part and by  $\bar{\mathfrak{G}}$  its closure ( $\infty$  belongs to  $\bar{\mathfrak{G}}$ ). Let  $\mathfrak{S}_n$  be the set of all symmetrical matrices  $\mathbf{Z}$  over  $R$  of type  $(n, n)$  which fulfil the condition

$$\operatorname{Re} \mathbf{x}^\top \mathbf{Z} \mathbf{x} \geq 0$$

for every real vector  $\mathbf{x}$  of type  $(n, 1)$  and for any  $p \in \mathfrak{G}$  which is not a pole of  $\mathbf{Z}$ . Let  $\mathfrak{P}_n$  be the set of all matrices belonging to  $\mathfrak{S}_n$  which fulfil the condition

$$\operatorname{Re} \mathbf{x}^\top \mathbf{Z} \mathbf{x} > 0$$

for every real nonzero vector  $\mathbf{x}$  of type  $(n, 1)$  and for every  $p \in \mathfrak{G}$  which is not a pole of  $\mathbf{Z}$ .

Obviously: a)  $\mathbf{Z}_1, \mathbf{Z}_2 \in \mathfrak{S}_n \Rightarrow \alpha_1 \mathbf{Z}_1 + \alpha_2 \mathbf{Z}_2 \in \mathfrak{S}_n$  provided  $\alpha_1, \alpha_2 \geq 0$ ,

b)  $\mathbf{Z}_1 \in \mathfrak{S}_n, \mathbf{Z}_2 \in \mathfrak{P}_n \Rightarrow \mathbf{Z}_1 + \mathbf{Z}_2 \in \mathfrak{P}_n$ ,

c) in particular, every positive (semi-)definite matrix belongs to  $(\mathfrak{S}_n) \mathfrak{P}_n$ .

**Theorem 3.** If  $\mathbf{Z} \in \mathfrak{S}_n$ , then there exist real numbers  $\omega_1, \dots, \omega_m$  and constant matrices  $\mathbf{H}_k \in \mathfrak{S}_n, k = 0, 1, \dots, m$ , such that

$$\mathbf{Z}(p) = \tilde{\mathbf{Z}}(p) + \mathbf{H}_0 p + \sum_{k=1}^m \mathbf{H}_k \frac{p}{p^2 + \omega_k^2},$$

where  $\tilde{\mathbf{Z}} \in \mathfrak{S}_n$  has no poles in  $\bar{\mathfrak{G}}$ .

**Theorem 4.** Let  $\mathbf{Z} \in \mathfrak{S}_n$ . Then  $\mathbf{Z} \in \mathfrak{P}_n$  if and only if  $\det \mathbf{Z} \neq 0$  for every  $p \in \mathfrak{G}$ .

**Theorem 5.** If  $\mathbf{Z} \in \mathfrak{P}_n$  then  $\mathbf{Z}^{-1}$  exists and  $\mathbf{Z}^{-1} \in \mathfrak{P}_n$ .

**Theorem 6.** If  $\mathbf{Z} \in \mathfrak{S}_n$  and  $\mathbf{C}$  is any real constant matrix of type  $(n, k)$ , then  $\mathbf{C}^\top \mathbf{Z} \mathbf{C} \in \mathfrak{S}_k$ .

Proofs of Theorems 3–6 can be found in [1], Chap. 4.

## 2. A CRITERION OF ASYMPTOTIC STABILITY OF PASSIVE NETWORK

Let  $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  be a passive network. The network  $N$  will be called *asymptotically stable* if for any real vectors  $\mathbf{i}_0, \mathbf{q}_0$  the solution  $\mathbf{i}$  of the network  $N$  corresponding to the vector  $\mathbf{e} = \mathbf{o}$  and initial conditions  $\mathbf{i}_0, \mathbf{q}_0$  exists and has no poles in  $\overline{\mathfrak{G}}$ .

Remark. If the conditions (K1), (K2) are interpreted as Laplace transforms of Kirchhoff's laws, then one can easily prove that for any solutions  $\mathbf{i}_1, \mathbf{i}_2$  of  $N$  corresponding to the same vector  $\mathbf{e}_0$  the difference  $\mathbf{i}_1 - \mathbf{i}_2$  (which is a solution of  $N$  corresponding to  $\mathbf{e} = \mathbf{o}$ ) has no poles in  $\overline{\mathfrak{G}}$  if and only if

$$\lim_{t \rightarrow \infty} \|\mathcal{L}^{-1}(\mathbf{i}_1)(t) - \mathcal{L}^{-1}(\mathbf{i}_2)(t)\| = 0.$$

**Theorem 7.** Let  $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  be a passive network. Suppose that the following conditions are fulfilled for every nonzero cycle  $\mathbf{c}^T \mathbf{v}$  of the graph  $G$ :

1.  $\mathbf{c}^T(\mathbf{R} + \mathbf{S}) \mathbf{c} > 0$ ,
2.  $\mathbf{c}^T(\mathbf{R} + \mathbf{L}) \mathbf{c} > 0$ ,
3. if  $\mathbf{c}^T \mathbf{R} \mathbf{c} = 0$  then there exists a nonzero cycle  $\tilde{\mathbf{c}}^T \mathbf{v}$  of  $G$  such that the conditions  $\tilde{\mathbf{c}}^T \mathbf{S} \mathbf{c} \neq 0$  and  $\tilde{\mathbf{c}}^T \mathbf{L} \mathbf{c} = 0$  are simultaneously fulfilled.

Then the network  $N$  is asymptotically stable.

Proof.

**Lemma 5.** Under the same assumptions as in Theorem 7,

$$\mathbf{W}(p) = \mathbf{X}^T \mathbf{Z}(p) \mathbf{X} \in \mathfrak{F}_n.$$

Proof. The network  $N$  is passive and hence by Theorem 6  $\mathbf{W} \in \mathfrak{S}_n$ . It follows from Theorem 4 that  $\mathbf{W} \in \mathfrak{F}_n$  if and only if  $\det \mathbf{W} \neq 0$  in  $\mathfrak{G}$ . Suppose that there exists  $p_0 \in \mathfrak{G}$  such that  $\det \mathbf{W}(p_0) = 0$ . Then there exists a nonzero vector  $\mathbf{w}$  such that  $\mathbf{W}(p_0) \mathbf{w} = \mathbf{o}$ , hence  $\operatorname{Re}(\mathbf{w}^T \mathbf{X}^T \mathbf{Z}(p_0) \mathbf{X} \mathbf{w}) = 0$ , which for  $p_0 = p'_0 + ip''_0$ ,  $\mathbf{c} = \mathbf{X} \mathbf{w}$  and nonzero cycle  $\mathbf{c}^T \mathbf{v}$  yields

$$(1) \quad \mathbf{c}^T \mathbf{R} \mathbf{c} + p'_0 \mathbf{c}^T \mathbf{L} \mathbf{c} + \frac{p'_0}{|p_0|^2} \mathbf{c}^T \mathbf{S} \mathbf{c} = 0.$$

By hypothesis, all terms on the left-hand side of (1) are non-negative and cannot be simultaneously zero, which is a contradiction.

**Lemma 6.** Under the same assumptions as in Theorem 7,

$$\det \mathbf{W}(i\omega_0) \neq 0$$

for every real  $\omega_0 \neq 0$ .

Proof. Suppose  $\det \mathbf{W}(i\omega_0) = 0$ ,  $\omega_0$  being a real nonzero number. Then there exists a real nonzero vector  $\mathbf{w}$  such that

$$(2) \quad \mathbf{W}(i\omega_0) \mathbf{w} = \mathbf{0}$$

and therefore for a nonzero cycle  $\mathbf{c}^T \mathbf{v}$ , where  $\mathbf{c} = \mathbf{X}\mathbf{w}$ , it holds

$$\mathbf{c}^T \mathbf{R} \mathbf{c} + i \left( \omega_0 \mathbf{c}^T \mathbf{L} \mathbf{c} - \frac{1}{\omega_0} \mathbf{c}^T \mathbf{S} \mathbf{c} \right) = 0$$

and hence  $\mathbf{c}^T \mathbf{R} \mathbf{c} = 0$ .

By assumption 3) of Theorem 7 there exists a cycle  $\tilde{\mathbf{c}}^T \mathbf{v}$  of  $G$  such that  $\tilde{\mathbf{c}}^T \mathbf{S} \mathbf{c} \neq 0$  and  $\tilde{\mathbf{c}}^T \mathbf{L} \mathbf{c} = 0$ . By Lemma 4 there exists a nonzero vector  $\tilde{\mathbf{w}}$  such that  $\tilde{\mathbf{c}} = \mathbf{X}\tilde{\mathbf{w}}$ . Then (2) implies  $\tilde{\mathbf{w}}^T \mathbf{W}(i\omega_0) \mathbf{w} = 0$ , consequently

$$\tilde{\mathbf{c}}^T \mathbf{R} \mathbf{c} + i \left( \omega_0 \tilde{\mathbf{c}}^T \mathbf{L} \mathbf{c} - \frac{1}{\omega_0} \tilde{\mathbf{c}}^T \mathbf{S} \mathbf{c} \right) = 0$$

and hence

$$\omega_0^2 \tilde{\mathbf{c}}^T \mathbf{L} \mathbf{c} = \tilde{\mathbf{c}}^T \mathbf{S} \mathbf{c}.$$

This contradiction proves our lemma.

**Lemma 7.** *Under the same assumptions as in Theorem 7 the matrix  $\mathbf{W}^{-1}$  has no poles in  $\overline{\mathfrak{G}}$ .*

Proof. Lemma 5 and Theorem 5 guarantee the existence of the matrix  $\mathbf{W}^{-1} \in \mathfrak{P}_n$ ; by Theorem 3,  $\mathbf{W}^{-1}$  has no poles in  $\overline{\mathfrak{G}}$  and the poles on the imaginary axis and at infinity are simple. Lemma 6 then implies that the only poles of  $\mathbf{W}^{-1}$  in  $\overline{\mathfrak{G}}$  can be 0 and  $\infty$ .

a) Suppose 0 is a pole of  $\mathbf{W}^{-1}$ . By Theorem 3 there exist matrices  $\mathbf{H}, \mathbf{K} \in \mathfrak{S}_n$  such that  $\mathbf{W}^{-1} = \mathbf{H}p^{-1} + \mathbf{K}(p)$ , where  $\mathbf{H}$  is a constant nonzero matrix and  $\mathbf{K}(p)$  has no pole in 0. Simultaneously

$$\mathbf{W}(p) = \mathbf{X}^T \mathbf{S} \mathbf{X} \frac{1}{p} + \mathbf{X}^T (\mathbf{R} + \mathbf{L}p) \mathbf{X}.$$

The obvious identity  $\mathbf{W}\mathbf{W}^{-1} = \mathbf{I}$  ( $\mathbf{I}$  is the unit matrix) then yields

$$\mathbf{I} = \mathbf{X}^T \mathbf{S} \mathbf{X} \mathbf{H} \frac{1}{p^2} + \mathbf{X}^T \mathbf{S} \mathbf{X} \mathbf{K}(p) \frac{1}{p} + \mathbf{X}^T (\mathbf{R} + \mathbf{L}p) \mathbf{X} \mathbf{H} \frac{1}{p} + \mathbf{X}^T (\mathbf{R} + \mathbf{L}p) \mathbf{X} \mathbf{K}(p).$$

This implies that  $\mathbf{X}^T \mathbf{S} \mathbf{X} \mathbf{H} = \mathbf{H} \mathbf{X}^T \mathbf{S} \mathbf{X} = \mathbf{0}$ . Multiplying by  $p$  and letting  $p \rightarrow 0$  one obtains

$$\mathbf{X}^T \mathbf{S} \mathbf{X} \mathbf{K}_0 + \mathbf{X}^T \mathbf{R} \mathbf{X} \mathbf{H} = \mathbf{0}$$

(where  $\mathbf{K}_0 = \lim_{p \rightarrow 0} \mathbf{K}(p)$ ). Consequently,  $\mathbf{H}\mathbf{X}^\top(\mathbf{R} + \mathbf{S})\mathbf{X}\mathbf{H} = \mathbf{0}$ . Suppose  $\mathbf{H}$  has a nonzero column  $\mathbf{h}$ . Then for a nonzero cycle  $\mathbf{c}^\top \mathbf{v} = (\mathbf{X}\mathbf{h})^\top \mathbf{v}$  of  $G$  one obtains

$$\mathbf{c}^\top(\mathbf{R} + \mathbf{S})\mathbf{c} = 0,$$

which contradicts assumption 1 of Theorem 7.

b) Suppose  $\infty$  is a pole of  $\mathbf{W}^{-1}$ . Similarly, from  $\mathbf{W}^{-1} = \mathbf{H}p + \mathbf{K}(p)$  and  $\mathbf{W} = \mathbf{X}^\top \mathbf{L} \mathbf{X} p + \mathbf{X}^\top(\mathbf{R} + \mathbf{S}p^{-1})\mathbf{X}$  one obtains  $\mathbf{H}\mathbf{X}^\top(\mathbf{R} + \mathbf{L})\mathbf{X}\mathbf{H} = \mathbf{0}$ , which contradicts assumption 2 of Theorem 7.

Proof of Theorem 7.

Let  $i(p)$  be a solution of  $N$  corresponding to the vector  $\mathbf{e} = \mathbf{0}$  and initial conditions  $i_0, \mathbf{q}_0$  (its existence follows from Theorem 2). By Theorem 1,

$$(3) \quad i(p) = \mathbf{A}(p) \begin{pmatrix} \mathbf{L}i_0 - \mathbf{S}\mathbf{q}_0 \\ \frac{1}{p} \end{pmatrix}$$

where

$$\mathbf{A}(p) = \mathbf{X}[\mathbf{X}^\top \mathbf{Z}(p)\mathbf{X}]^{-1} \mathbf{X}^\top = \mathbf{X}\mathbf{W}^{-1} \mathbf{X}^\top.$$

From Lemma 7 it follows that  $\mathbf{A}$  has no poles in  $\overline{\mathbb{C}}$  and hence the only pole of  $i$  in  $\overline{\mathbb{C}}$  can be 0.

Suppose 0 is a pole of  $\mathbf{W}^{-1} \mathbf{X}^\top \mathbf{S} p^{-1}$ . Then there exist matrices  $\mathbf{H}, \mathbf{K}$  of type  $(n, r)$  such that

$$\mathbf{W}^{-1} \mathbf{X}^\top \mathbf{S} p^{-1} = \mathbf{H}p^{-1} + \mathbf{K}(p),$$

where  $\mathbf{H}$  is a constant matrix and  $\mathbf{K}(p)$  is regular at 0 (and hence  $\mathbf{K}_0 = \lim_{p \rightarrow 0} \mathbf{K}(p)$  exists). This implies further that

$$\mathbf{X}^\top \mathbf{S} p^{-1} = \mathbf{W}(\mathbf{H}p^{-1} + \mathbf{K}(p)),$$

which yields

$$\begin{aligned} \mathbf{X}^\top \mathbf{S} p^{-1} &= \mathbf{X}^\top \mathbf{L} \mathbf{X} \mathbf{K}(p) p + \mathbf{X}^\top \mathbf{L} \mathbf{X} \mathbf{H} + \mathbf{X}^\top \mathbf{R} \mathbf{X} \mathbf{K} + (\mathbf{X}^\top \mathbf{R} \mathbf{X} \mathbf{H} + \mathbf{X}^\top \mathbf{S} \mathbf{X} \mathbf{K}) p^{-1} + \\ &\quad + \mathbf{X}^\top \mathbf{S} \mathbf{X} \mathbf{H} p^{-2}. \end{aligned}$$

This implies that  $\mathbf{X}^\top \mathbf{S} \mathbf{X} \mathbf{H} = \mathbf{0}$  and therefore

$$(4) \quad \mathbf{H}^\top \mathbf{X}^\top \mathbf{S} \mathbf{X} \mathbf{H} = \mathbf{0}.$$

Multiplying by  $p$  and letting  $p \rightarrow 0$  one obtains  $\mathbf{X}^\top \mathbf{S} = \mathbf{X}^\top \mathbf{R} \mathbf{X} \mathbf{H} + \mathbf{X}^\top \mathbf{S} \mathbf{X} \mathbf{K}_0$  and hence

$$(5) \quad \mathbf{H}^\top \mathbf{X}^\top \mathbf{S} = \mathbf{H}^\top \mathbf{X}^\top \mathbf{R} \mathbf{X} \mathbf{H}.$$

Suppose that the  $j$ -th column  $\mathbf{h}$  of  $\mathbf{H}$  is nonzero. Then  $\mathbf{d}^\top \mathbf{v} = (\mathbf{X}\mathbf{h})^\top \mathbf{v}$  is a nonzero cycle of  $G$  and it follows from (4) that  $\mathbf{d}^\top \mathbf{S} \mathbf{d} = 0$ , therefore by assumption 1  $\mathbf{d}^\top \mathbf{R} \mathbf{d} > 0$

and hence the element  $(j, j)$  of the matrix  $\mathbf{H}^T \mathbf{X}^T \mathbf{R} \mathbf{X} \mathbf{H}$  is nonzero. However, from  $\mathbf{d}^T \mathbf{S} \mathbf{d} = 0$  and from the fact that  $\mathbf{S}$  is a diagonal positive semidefinite matrix it follows that  $\mathbf{d}^T \mathbf{S} = \mathbf{0}$ , and hence the  $j$ -th row of the matrix  $\mathbf{H}^T \mathbf{X}^T \mathbf{S}$  is zero, which contradicts (5). This contradiction proves that  $\mathbf{W}^{-1} \mathbf{X}^T \mathbf{S} \mathbf{p}^{-1}$  has no poles in  $\bar{\mathcal{U}}$  and it follows from (3) that  $\mathbf{i}$  has the same property.

From Theorem 7 one can immediately obtain the following well-known theorem:

**Theorem 8.** *Let  $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  be a passive network. Suppose  $\mathbf{d}^T \mathbf{R} \mathbf{d} > 0$  for each loop  $\mathbf{d}^T \mathbf{v}$  of  $G$ . Then  $N$  is asymptotically stable.*

Proof follows from Theorem 7, Lemma 2 and from the diagonality of  $\mathbf{R}$ .

For networks with a diagonal matrix  $\mathbf{L}$  one can obtain the following

**Theorem 9.** *Let  $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$  be a passive network with a diagonal matrix  $\mathbf{L}$ . Suppose the following conditions are fulfilled for every nonzero loop  $\mathbf{d}^T \mathbf{v}$  of  $G$ :*

1.  $\mathbf{d}^T (\mathbf{R} + \mathbf{S}) \mathbf{d} > 0$ ,
2.  $\mathbf{d}^T (\mathbf{R} + \mathbf{L}) \mathbf{d} > 0$ ,
3. if  $\mathbf{d}^T \mathbf{R} \mathbf{d} = 0$ , then there exists a loop  $\tilde{\mathbf{d}}^T \mathbf{v}$  of  $G$  such that simultaneously  $\tilde{\mathbf{d}}^T \mathbf{S} \mathbf{d} \neq 0$  and  $\tilde{\mathbf{d}}^T \mathbf{L} \mathbf{d} = 0$ .

Then the network  $N$  is asymptotically stable.

Proof. Theorem 9 can be proved in a similar manner as Theorem 7. By Lemma 2, assumptions 1 and 2 of Theorem 9 are equivalent with those of Theorem 7. Assumption 3 is used only in the proof of Lemma 6, which can be proved analogously using assumption 3 of Theorem 9, Lemma 1 and the diagonality of the matrices  $\mathbf{R}, \mathbf{L}, \mathbf{S}$ .

Remark. From the physical view-point, Theorem 9 gives sufficient conditions of asymptotic stability which can be used for networks with loops without nonzero resistors. Such a loop without nonzero resistors must contain a nonzero capacitor and an inductor (assumptions 1 and 2) and the capacitor must be contained in another loop (assumption 3). Theorem 7 is a generalization of this condition to networks with inductive couplings.

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## Souhrn

# O ASYMPTOTICKÉ STABILITĚ PASIVNÍCH LINEÁRNÍCH ELEKTRICKÝCH OBVODŮ

ZDENĚK RYJÁČEK

V práci je uvedeno kritérium asymptotické stability řešení lineárního elektrického obvodu se soustředěnými parametry, jež je oslabením podmínek dosud známých — kritérium lze použít i na obvody, jejichž některé smyčky neobsahují nenulový ohmický odpor.

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