Graham Smith; Jeremy Dawson

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THE DETERMINATION OF NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A SOLUTION TO THE 3 × 3 × 3 MULTI-INDEX PROBLEM

GRAHAM SMITH, JEREMY DAWSON

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INTRODUCTION

In a previous paper Smith [4] gave a procedure for determining necessary and sufficient conditions for the existence of a feasible solution to the multi-index problem: maximize:

$$z = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} c_{ijk} \cdot x_{ijk}$$

subject to:

$$\sum_{k=1}^{n} x_{ijk} = A_{ij} \quad (i = 1, 2, \ldots, l;\ j = 1, 2, \ldots, m) \tag{1.1}$$

$$\sum_{j=1}^{m} x_{ijk} = B_{ik} \quad (i = 1, 2, \ldots, l;\ k = 1, 2, \ldots, n) \tag{1.2}$$

$$\sum_{i=1}^{l} x_{ijk} = C_{jk} \quad (j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n) \tag{1.3}$$

where:

$$x_{ijk} \geqq 0 \quad (i = 1, 2, \ldots, l;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n) \tag{1.4}$$

$$\sum_{i=1}^{l} A_{ij} = \sum_{k=1}^{n} C_{jk} \quad (j = 1, 2, \ldots, m) \tag{2.1}$$

$$\sum_{j=1}^{m} C_{jk} = \sum_{i=1}^{l} B_{ik} \quad (k = 1, 2, \ldots, n) \tag{2.2}$$

$$\sum_{k=1}^{n} B_{ik} = \sum_{j=1}^{m} A_{ij} \quad (i = 1, 2, \ldots, l)\,. \tag{2.3}$$

It was shown that the above problem has no feasible solution if and only if the problem
minimize:

$$(3)\qquad \phi' = \{\sum_{i=1}^{l}\sum_{k=1}^{n} B_{ik} - \sum_{i=2}^{l} A_{i1} - \sum_{k=2}^{n} B_{1k} - \sum_{j=2}^{m} C_{j1}\}\,\alpha' +$$

$$+ \sum_{i=2}^{l}\{B_{i1} - \sum_{j=2}^{m} A_{ij}\}\,\beta_i' + \sum_{j=2}^{m}\{A_{1j} - \sum_{k=2}^{n} C_{jk}\}\,\gamma_j' +$$

$$+ \sum_{k=2}^{n}\{C_{1k} - \sum_{i=2}^{l} B_{ik}\}\,\delta_k' + \sum_{i=2}^{l}\sum_{j=2}^{m} A_{ij}\xi_{ij}' + \sum_{i=2}^{l}\sum_{k=2}^{n} B_{ik}\eta_{ik}' + \sum_{j=2}^{m}\sum_{k=2}^{n} C_{jk}\theta_{jk}'$$

subject to:

$$(4.1)\qquad \alpha' \geqq 1$$

$$(4.2)\qquad \beta_i' \geqq 1 \quad (i = 2, 3, \ldots, l)$$

$$(4.3)\qquad \gamma_j' \geqq 1 \quad (j = 2, 3, \ldots, m)$$

$$(4.4)\qquad \delta_k' \geqq 1 \quad (k = 2, 3, \ldots, n)$$

$$(4.5)\qquad \xi_{ij}' \geqq 1 \quad (i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m)$$

$$(4.6)\qquad \eta_{ik}' \geqq 1 \quad (i = 2, 3, \ldots, l;\ k = 2, 3, \ldots, n)$$

$$(4.7)\qquad \theta_{jk}' \geqq 1 \quad (j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n)$$

$$(4.8)\qquad \alpha' - \beta_i' - \gamma_j' - \delta_k' + \xi_{ij}' + \eta_{ik}' + \theta_{jk}' \geqq 1$$

$$(i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n)$$

is unbounded. It was further shown that the dual to (3,4) has an objective function value equal to $\sum_{i=1}^{l}\sum_{j=1}^{m} A_{ij}$ for all feasible solutions, and it therefore follows that problem (3, 4) is unbounded (and therefore problem (1) has no feasible solution) if and only if there exists a feasible solution to (3, 4) such that $\varphi' < \sum_{i=1}^{l}\sum_{j=1}^{m} A_{ij}$.

The procedure for determining necessary and sufficient conditions for the existence of a solution to problem (1) entailed constructing every feasible simplex tableau for problem (3, 4). This proved to be so consuming of computer time that even a problem having $l = m = n = 3$ could not be solved.

Therefore the possibility of improving the procedure was investigated. In this paper, modifications to the procedure which considerably reduce the computer time requirements will be given. Furthermore results of applying the improved procedure to a problem having $l = m = n = 3$ will be presented. This is the smallest problem for which necessary and sufficient conditions for the existence of a solution were not previously known.

The first improvement is to reduce the tableau size by noting that the constraints (4.1), (4.2), ..., (4.7) are merely lower bounds. These constraints can therefore be eliminated by the standard linear programming transformation for lower bounded variables (see for example Vajda [6] page 105). This gives rise to the linear programming problem
minimize:

$$(5) \qquad \phi = \{\sum_{i=1}^{l}\sum_{k=1}^{n} B_{ik} - \sum_{i=2}^{l} A_{i1} - \sum_{k=2}^{n} B_{1k} - \sum_{j=2}^{m} C_{j1}\}\,\alpha + \\ + \sum_{i=2}^{l}\{B_{i1} - \sum_{j=2}^{m} A_{ij}\}\,\beta_i + \sum_{j=2}^{m}\{A_{1j} - \sum_{k=2}^{n} C_{jk}\}\,\gamma_j + \\ + \sum_{k=2}^{n}\{C_{1k} - \sum_{i=2}^{l} B_{ik}\}\,\delta_k + \sum_{i=2}^{l}\sum_{j=2}^{m} A_{ij}\xi_{ij} + \sum_{i=2}^{l}\sum_{k=2}^{n} B_{ik}\eta_{ik} + \sum_{j=2}^{m}\sum_{k=2}^{n} C_{jk}\,.\,\theta_{jk}$$

subject to:

$$(6) \qquad \alpha - \beta_i - \gamma_j - \delta_k + \xi_{ij} + \eta_{ik} + \theta_{jk} \geqq 0$$
$$(i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n)$$

where

$$(7) \qquad \alpha, \beta_i, \gamma_j, \delta_k, \xi_{ij}, \eta_{ij}, \theta_{jk} \geqq 0$$
$$(i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n)$$

$$(8.1) \qquad \alpha = \alpha' - 1$$
$$(8.2) \qquad \beta_i = \beta_i' - 1 \quad (i = 2, 3, \ldots, l)$$
$$(8.3) \qquad \gamma_j = \gamma_j' - 1 \quad (j = 2, 3, \ldots, m)$$
$$(8.4) \qquad \delta_k = \delta_k' - 1 \quad (k = 2, 3, \ldots, n)$$
$$(8.5) \qquad \xi_{ij} = \xi_{ij}' - 1 \quad (i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m)$$
$$(8.6) \qquad \eta_{ij} = \eta_{ik}' - 1 \quad (i = 2, 3, \ldots, l;\ k = 2, 3, \ldots, n)$$
$$(8.7) \qquad \theta_{jk} = \theta_{jk}' - 1 \quad (j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n)\,.$$

Lemma 1. *Problem* (1) *has no feasible solution if and only if there exists a feasible solution to problem* (5, 6, 7) *such that* $\phi < 0$.

Proof. The equations (8) define a one to one correspondence between solutions to problem (3,4) and solutions to problem (5, 6, 7). Using (2), (3), (5) and (8) it can be shown that

$$\phi' - \phi = \sum_{i=1}^{l}\sum_{j=1}^{m} A_{ij}\,.$$

Since problem (1) has no feasible solution if and only if there exists a feasible solution to problem (3, 4) such that

$$\phi' < \sum_{i=1}^{l} \sum_{j=1}^{m} A_{ij}$$

it therefore follows that problem (1) has no feasible solution if and only if there exists a feasible solution to problem (5, 6, 7) such that $\phi < 0$.

Since problem (5, 6, 7) has fewer rows than problem (3, 4), less computation is required for simplex tableaux transformations with the former than with the latter.

GENERATING CONDITIONS MORE DIRECTLY

The procedure [4] as originally described involves inspecting every element of every feasible simplex tableaux of problem (3,4) in order to discover every non-positive column vector of coefficients which indicates unboundedness. Therefore the computation involved in constructing tableaux which do not contain such vectors (or which only contain previously encountered vectors of this type) is in a sense wasted. The second improvement aims to avoid constructing such tableaux unnecessarily, by adding to problem (5, 6, 7) the constraint:

$$\alpha + \sum_{i=2}^{l} \beta_i + \sum_{j=2}^{m} \gamma_j + \sum_{k=2}^{n} \delta_k + \sum_{i=2}^{l} \sum_{j=2}^{m} \xi_{ij} + \sum_{i=2}^{l} \sum_{k=2}^{n} \eta_{ik} + \sum_{j=2}^{m} \sum_{k=2}^{n} \theta_{jk} = 1 \tag{9}$$

to form a new linear programming problem (5, 6, 7, 9). It will be shown that a necessary condition for the existence of a solution to problem (1) can be constructed directly from each basic feasible solution to (5, 6, 7, 9) thus avoiding the complete inspection of each tableau.

Lemma 2. *If there exists a feasible solution to problem* (5, 6, 7) *such that* $\phi < 0$, *then there exists a basic feasible solution to problem* (5, 6, 7, 9) *such that* $\phi < 0$

Proof. Consider a feasible solution to problem (5, 6, 7) such that $\phi < 0$. A least one of the variables must be non-zero in this solution since otherwise $\phi = 0$. Scale this feasible solution by dividing the value of each variable by the non-zero quantity

$$\left(\alpha + \sum_{i=2}^{l} \beta_i + \sum_{j=2}^{m} \gamma_j + \sum_{k=2}^{n} \delta_k + \sum_{i=2}^{l} \sum_{j=2}^{m} \xi_{ij} + \sum_{i=2}^{l} \sum_{k=2}^{n} \eta_{ik} + \sum_{j=2}^{m} \sum_{k=2}^{n} \theta_{jk}\right).$$

The resulting point satisfies (6, 7, 9) with $\phi < 0$, and thus the optimal solution to problem (5, 6, 7, 9) has $\phi < 0$. This optimal solution is basic since the objective function (5) is bounded because (9) implies an upper bound of 1 on each variable. Therefore there exists a basic feasible solution to (5, 6, 7, 9) such that $\phi < 0$.

Lemma 3. *If there exists a basic feasible solution to problem* (5, 6, 7, 9) *such that* $\phi < 0$ *then problem* (5, 6, 7) *has a feasible solution such that* $\phi < 0$.

Proof. This is immediately evident since any point satisfying the constraints (6, 7, 9) also satisfies the constraints (6, 7).

Theorem 1 given below is a direct consequence of Lemmas 1, 2 and 3.

Theorem 1. *Problem* 1 *has no feasible solution if and only if there exists a basic feasible solution to the linear programming problem* (5, 6, 7, 9) *such that* $\phi < 0$.

NEW PROCEDURE FOR DETERMINING NECESSARY AND SUFFICIENT CONDITIONS

Theorem 1 provides the basis for a new procedure for determining necessary and sufficient conditions for the existence of a solution to the multi-index problem. This new procedure entails finding each vertex of the constraint set (6, 7, 9) and determining the condition for ϕ to be non-negative at that vertex.

There are several published methods for searching for all of the vertices of a set of linear constraints (see for example Balinski [1], Manas and Nedoma [2], or Mattheiss [3]).

For each vertex the values of the variables are substituted into (5) which gives an expression which must be non-negative for a solution to exist. The complete set of such expressions comprises the necessary and sufficient conditions for the existence of a solution to the multi-index problem.

The new procedure was programmed for an IBM 360/50 computer using the method of Balinski to search for the complete set of vertices of the constraint set (6, 7, 9).

A further improvement to the procedure which takes advantage of the symmetry of the problem will now be described and then some results of using the procedure will be given.

USING SYMMETRY

The computer search to find all vertices of the feasible region to problem (6, 7, 9) can be shortened by using the symmetry of the problem. Two types of symmetry between vertices will be demonstrated.

Given a vertex of this feasible region

$$V = (\alpha, \beta_i, \gamma_j, \delta_k, \xi_{ij}, \eta_{ik}, \theta_{jk}),$$

then

$$W = (\alpha, \beta'_i, \gamma'_j, \delta'_k, \xi'_{ij}, \eta'_{ik}, \theta'_{jk})$$

is also a vertex

where:

$$\beta'_i = \beta_{\pi(i)}, \qquad \gamma'_j = \gamma_{\varrho(j)}, \quad \delta'_k = \delta_{\sigma(k)},$$

$$\xi'_{ij} = \xi_{\pi(i)\varrho(j)}, \quad \eta'_{ik} = \eta_{\pi(i)\sigma(k)}, \quad \theta'_{jk} = \theta_{\varrho(j)\sigma(k)}$$

$$(i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n),$$

and π, ϱ and σ are permutations of $\{2, 3, \ldots, l\}$ $\{2, 3, \ldots, m\}$ and $\{2, 3, \ldots, n\}$ respectively.

This will be referred to subsequently as a symmetry of the first type.

Also if $l = m$, W is a vertex where:

$$\beta'_i = \gamma_i, \quad \gamma'_j = \beta_j, \quad \delta'_k = \delta_k, \quad \xi'_{ij} = \xi_{ji}, \quad \eta'_{ik} = \theta_{ik}. \quad \theta'_{jk} = \eta_{jk}$$

$$(i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n).$$

Furthermore if $l = m = n$, W is a vertex in each of the following cases:

$$\beta'_i = \delta_i, \quad \gamma'_j = \gamma_j, \quad \delta'_k = \beta_k, \quad \xi'_{ij} = \theta_{ji}, \quad \eta'_{ik} = \eta_{ki}, \quad \theta'_{jk} = \xi_{kj};$$

$$\beta'_i = \beta_i, \quad \gamma'_j = \delta_j, \quad \delta'_k = \gamma_k, \quad \xi'_{ij} = \eta_{ij}, \quad \eta'_{ik} = \xi_{ik}, \quad \theta'_{jk} = \theta_{kj};$$

$$\beta'_i = \delta_i, \quad \gamma'_j = \beta_j, \quad \delta'_k = \gamma_k, \quad \xi'_{ij} = \eta_{ji}, \quad \eta'_{ik} = \theta_{ki}, \quad \theta'_{jk} = \xi_{jk};$$

$$\beta'_i = \gamma_i, \quad \gamma'_j = \delta_j, \quad \delta'_k = \beta_k, \quad \xi'_{ij} = \theta_{ij}, \quad \eta'_{ik} = \xi_{ki}, \quad \theta'_{jk} = \eta_{kj};$$

$$(i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n).$$

This will be referred to subsequently as a symmetry of the second type.

The new procedure for generating necessary and sufficient conditions was applied to a multi-index problem having $l = m = n = 3$. The computer search for vertices was limited to those vertices on the hyperplane $\theta_{33} = 0$. Then the list of vertices found was extended as far as possible by the symmetries described above to generaet vertices having $\theta_{33} \neq 0$. It is now necessary to show that this procedure finds all the vertices of the feasible region.

Given a vertex V in which $\theta_{33} \neq 0$, V has nine basic variables and therefore at most nine of the variables are non-zero. Since the variables ξ_{ij}, η_{ik}, θ_{jk} number twelve, one (at least) must be non-zero, and so V can be transformed by a symmetry of the first type to a vertex V' in which $\xi_{33} = 0$ or $\eta_{33} = 0$ or $\theta_{33} = 0$. Then V' can be transformed by a symmetry of the second type to a vertex V'' with $\theta_{33} = 0$.

Now such a vertex V'' (which would be found by the computer search because it is on the hyperplane $\theta_{33} = 0$) can be transformed to the given vertex V by the inverse transformation, a symmetry of the second type followed by a symmetry of the first type.

Hence by finding all vertices V'' with $\theta_{33} = 0$, and taking all vertices derived from them by symmetries of the two types specified above, all vertices are generated.

Smith [5] has conjectured that the necessary conditions:

$$-\sum_{(i,j)\in\lambda} A_{ij} + \sum_{(i,k)\in\mu} B_{ik} + \sum_{(j,k)\in\nu} C_{jk} \geqq 0 \quad \text{all} \quad \lambda, \mu, \nu \tag{10}$$

(where $L = \{1, 2, \ldots, l\}$; $M = \{1, 2, \ldots, m\}$; $N = \{1, 2, \ldots, n\}$; $\lambda \subseteq L \times M$; $\mu \subseteq L \times N$; $\nu \subseteq M \times N$ such that if $(i, j) \in \lambda$, then for each k, either $(i, k) \in \mu$ or $(j, k) \in \nu$) are sufficient for the existence of a solution to multi-index problems having $l = m = n = 3$.

This conjecture was substantiated by applying the procedure described in this paper. The conditions generated by the procedure were rearranged using the equations (2), yielding the equivalent conditions:

$$\begin{aligned}
&-A_{i_1j_3} + B_{i_1k_1} + B_{i_1k_2} + C_{j_3k_3} \geqq 0 \quad \text{all} \quad i_1, j_3, k_1, k_2\\
&-A_{i_1j_3} + B_{i_1k_1} + C_{j_3k_2} + C_{j_3k_3} \geqq 0 \quad \text{all} \quad i_1, j_3, k_1, k_2\\
&-A_{i_1j_3} - A_{i_2j_3} + B_{i_1k_1} + B_{i_2k_1} + B_{i_1k_2} + B_{i_2k_2} + C_{j_3k_3} \geqq 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{all} \quad i_1, i_2, j_3, k_1, k_2\\
&-A_{i_1j_3} - A_{i_2j_3} + B_{i_1k_1} + B_{i_2k_1} + C_{j_3k_2} + C_{j_3k_3} \geqq 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{all} \quad i_1, i_2, j_3, k_1, k_2\\
&-A_{i_1j_1} - A_{i_1j_2} - A_{i_2j_2} + B_{i_1k_1} + B_{i_2k_1} + B_{i_1k_2} + C_{j_2k_2} + C_{j_1k_3} + C_{j_2k_3} \geqq 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{all} \quad i_1, i_2, j_2, k_1, k_2\\
&B_{i_1k_1}, \quad C_{j_3k_1}, \quad A_{i_1j_3} \geqq 0 \qquad\qquad \text{all} \quad i_1, j_3, k_1
\end{aligned}$$

where:

$$i_1, i_2, i_3, j_3, k_1, k_2, k_3 \in \{1, 2, 3\}$$
$$j_1, j_2, \in \{1, 2\}$$

such that:

$$i_1 \neq i_2, \quad i_1 \neq i_3, \quad i_2 \neq i_3, \quad k_1 \neq k_2, \quad k_1 \neq k_3, \quad k_2 \neq k_3, \quad j_1 \neq j_2 .$$

It can be shown that these generated necessary and sufficient conditions are a proper subset of the conditions (10) and therefore the conditions (10) are necessary and sufficient for the existence of a solution to multi-index problems having $l = m = n = 3$.

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Souhrn

URČENÍ NUTNÝCH A POSTAČUJÍCÍCH PODMÍNEK EXISTENCE ŘEŠENÍ PROBLÉMU 3 × 3 × 3 MULTIINDEXU

GRAHAM SMITH, JEREMY DAWSON

Jsou popsány úpravy procesu určení nutných a postačujících podmínek existence řešení problému multiindexu. Tyto úpravy redukují potřebné výpočty do té míry, že je možno určit nutné a postačující podmínky existence pro problém 3 × 3 × 3 multiindexu. Tyto podmínky jsou v práci uvedeny.

Authors' addresses: Dr. *Graham Smith*, Dr. *Jeremy Dawson*, Department of Industrial Engineering, The University of New South Wales, P. O. Box 1, Kensington, N. S. W. 2033.