Peter Růžička; Juraj Wiedermann

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Aplikace matematiky, Vol. 24 (1979), No. 5, 321--325

Persistent URL: http://dml.cz/dmlcz/103813

ON THE CONJECTURE RELATING MINIMAX AND MINIMEAN COMPLEXITY NORMS

PETER RUŽIČKA, JURAJ WIEDERMANN

(Received June 28, 1977)

1. INTRODUCTION

Only relatively few algorithms are known to be worst-case or minimax optimal among comparison problems. Still fewer are known to be minimean optimal, that is minimizing the average number of comparisons assuming random order.

From widely known algorithms minimizing the average number of comparisons, we mention the algorithms for finding the maximum element and for selecting both the maximum and the minimum element from an unordered set [1].

Both the above minimean optimal algorithms have been proved to be minimax optimal. The circumstance that these algorithms are of uniform complexity (they perform an identical number of comparisons for all input permutations, and so their minimax and minimean complexities are equal to each other) leads to an intuitive idea formulated by Ira Pohl [2, 3] in the form of the following conjecture:

A minimax optimal algorithm is also minimean optimal if

1. it is minimean optimal over all minimax optimal algorithms,
2. it has uniform complexity.

Let $V_k(n)$ or $\bar{V}_k(n)$ ($\bar{V}_k^{\max}(n)$) be the minimum number of comparisons sufficient for selecting the k-th largest element of an n-element set in the minimax or the minimean case (over all minimax optimal algorithms), respectively.

Especially when searching for $V_k(n)$, the nonexistence of an algorithm optimal both in the minimax and the minimean case has not been proved. It is well known that such algorithms do exist for $k = 2$ and $n = 3, 4, 5$ [1]. Knuth guessed that $V_2(6)$ could be the first problem in which a minimean optimal algorithm is not minimax optimal. He started from the familiar facts that $\bar{V}_2^{\max}(6) \leqq 6\ 2/3$ and $\bar{V}_2(6) \leqq 6\ 1/2$. Pohl [3] constructed an algorithm over all minimax algorithms doing $6\ 26/45$ comparisons in average, that is $\bar{V}_2^{\max}(6) \leqq 6\ 26/45$. Using a computer, the authors showed that $\bar{V}_2^{\max}(6) = 6\ 26/45$ and validated Knuth's conjecture that $\bar{V}_2(6) < \bar{V}_2^{\max}(6)$.

In the next section we show that a similar case occurs also when computing the median of five elements, that is $\bar{V}_3(5) < \bar{V}_3^{\max}(5)$. Furthermore, the minimean optimal algorithm over all minimax optimal algorithms has uniform complexity. These facts contradict Pohl's conjecture relating two complexity norms.

2. COUNTEREXAMPLE

Let X be a set $\{a_1, \ldots, a_n\}$. By the symbol $k\theta X$ we mean the k-th largest element of X. The crucial comparison for $a \in X$, $a \neq k\theta X$, is the first comparison $a : b$ such that either $b = k\theta X$ or $a < b < k\theta X$ or $k\theta X < b < a$. For each element $\neq k\theta X$ its relation to $k\theta X$ must be known. Hence the following lemma holds:

Lemma 1. *Each algorithm for selecting $k\theta X$ performs precisely $n - 1$ crucial comparisons.*

Lemma 2. $V_2(4) = \bar{V}_2(4) = 4$.

It is known that $V_3(5) = 6$. We are looking for a minimean optimal algorithm doing uniformly 6 comparisons.

Theorem 1. $\bar{V}_3^{\max}(5) = 6$.

Proof. Consider the class of all minimax algorithms for selecting the median out of 5 elements. Excluding symmetrical cases, there are only two possible starting comparisons

1. $a_1 < a_2$, $a_3 < a_2$
2. $a_1 < a_2$, $a_3 < a_4$

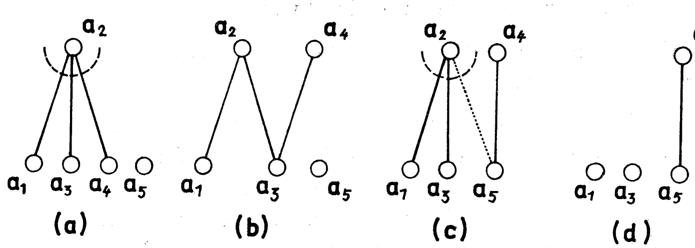

Fig. 1.

1a. In case $a_2 > a_4$ we see from the diagram (a) in Fig. 1 that a_2 cannot be the median and that the median is $2\theta\{a_1, a_3, a_4, a_5\}$, i.e. the second greatest element from the set where no relations are known. For determining $V_2(4)$ we need four comparisons, therefore case 1a requires 7 comparisons in the worst case.

1b. In case $a_4 > a_3$ we get the diagram (b) in Fig. 1 and assuming that a_5 is the median and $a_4 > a_5$, then all three comparisons are noncrucial. We know from Lemma 1 that at least 4 crucial comparisons have to be made; again 7 comparisons are needed.

1c. In case $a_4 > a_5$ we obtain the diagram (c) in Fig. 1;

i) when $a_2 > a_5$, a_2 cannot be the median, and to find the second largest element of the resulting structure (d) in Fig. 1 three more comparisons are needed by Lemma 2. This again gives 7 comparisons.

ii) when $a_3 < a_4$ the argument in 1b can be employed.

iii) case $a_2 > a_4$ can be reduced to 1ci.

iv) when $a_3 > a_5$, neither a_2 nor a_5 can be the median and $V_2(3) = 3$, which leads to 7 comparisons.

2a. Case $a_1 < a_4$ leads to the same case as in 1b.

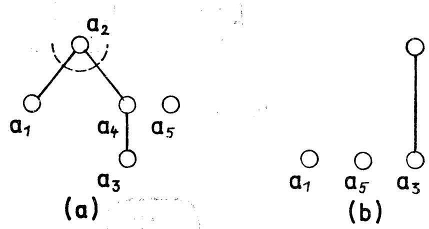

Fig. 2.

2b. In case $a_2 > a_4$ we get the diagram (a) in Fig. 2, where a_2 cannot be the median. Therefore the second largest element of the structure (b) in Fig. 2 is to be found. Using Lemma 2 we need 3 additional comparisons. The uniform complexity of the minimean optimal algorithm in $V_2(4)$ guarantees the uniform complexity of our algorithm, which makes at the worst 6 comparisons.

2c. Case $a_2 > a_5$ leads to 1c.

From the above proof we can easily construct the minimean optimal algorithm over all minimax optimal ones for determining the median out of five elements (Fig. 3).

Each "symmetrical" branch is identical to its brother, with indices of compared elements permuted in an appropriate manner. External nodes contain the index of the median element and the number of permutations leading to an external node appears immediately below it. This decision tree corresponds to the standard Hadian-Sobel method [1] for determining $V_k(n)$.

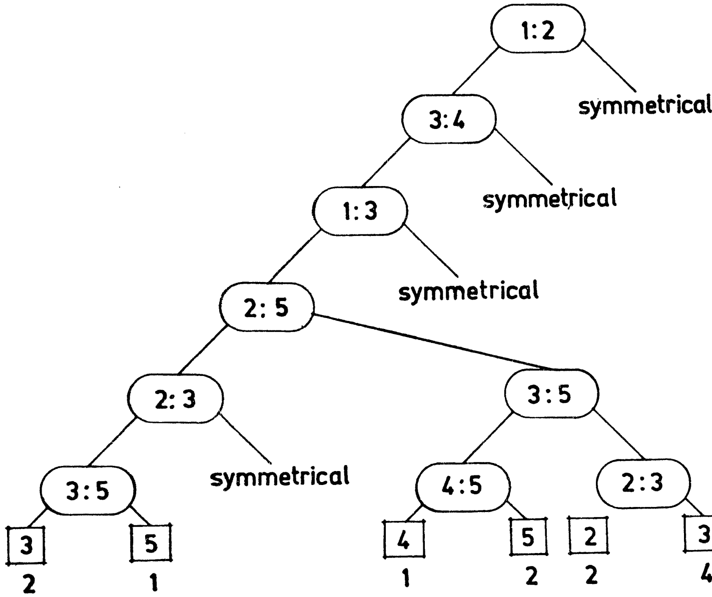
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Fig. 3.

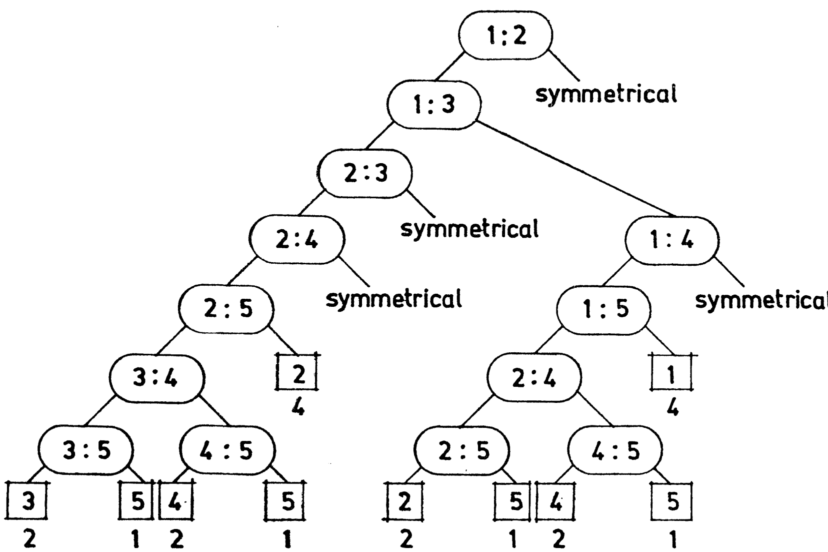
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Fig. 4.

Theorem 2. $\bar{V}_3(5) < \bar{V}_3^{\max}(5)$.

Proof. It is sufficient to find an algorithm doing less than 6 comparison in average. The decision tree in Fig. 4 corresponds to an algorithm with the average complexity 5 13/15.

It is interesting that this decision tree corresponds to the special case of Floyd's general algorithm for determining $k\theta X$ [1] which is conjectured by Floyd to be minimean optimal only assymptotically.

Theorems 1 and 2 contradict the validity of Pohl's conjecture concerning two conditions a minimax optimal algorithm must meet to be minimean optimal.

References

[1] *D. E. Knuth:* The Art of Computer Programming. Volume 3. Sorting and Searching. Addison-Wesley. 1973.

[2] *I. Pohl:* Minimean Optimality in Sorting Algorithms. Proceedings of 16th Annual Symposium on Foundations of Computer Science. 1975, 71—74.

[3] *I. Pohl:* On Selection over Six Elements and a Conjecture Relating Two Complexity Norms. Vrije Universiteit. Wiskundik Seminarium. Amsterdam. March 1975.

Súhrn

O DOMNIENKE, DÁVAJÚCEJ DO SÚVISLOSTI DVE MIERY ZLOŽITOSTI

PETER RUŽIČKA, JURAJ WIEDERMANN

V článku sa pomocou kontrapríkladu ukazuje, že optimálnosť v priemernom prípade spolu s rovnomernou zložitosťou rozhodovacích algoritmov, vybraných z triedy algoritmov optimálnych v najhoršom prípade, nestačí na to, aby tieto algoritmy boli optimálne v priemernom prípade v triede všetkých rozhodovacích algoritmov.

Author's addresses: RNDr. *Peter Ružička,* RNDr. *Juraj Wiedermann,* Výskumné výpočtové stredisko, Dúbravská cesta 3, 885 31 Bratislava.