Aplikace matematiky

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Aplikace matematiky, Vol. 24 (1979), No. 6, 421--426

Persistent URL: http://dml.cz/dmlcz/103825

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SOME REMARKS ON NUMERICAL SOLUTION OF INITIAL PROBLEMS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS

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(Received December 27, 1976)

1. INTRODUCTION

The subject of this paper is numerical solution of systems of differential equations

$$(1) \qquad y'(t) = f(t, y(t)), \quad y(a) = \eta,$$

for $t \in I = [a, b]$. We suppose that the continuous function $f : I \times R^m \to R^m$ satisfies the condition

$$(2) \qquad \|f(t, y_1) - f(t, y_2)\| \leqq \omega(t, \|y_1 - y_2\|)$$

for (t, y_1), $(t, y_2) \in R^m$ where the continuous function $\omega : I \times R^1_+ \to R^1_+ = [0, \infty)$ satisfies adequate conditions (see Assumption A).

This system has a unique solution by a theorem of [2]. This unique solution of (1) will be denoted by y throughout the paper. The solution y is approximated by using difference equations.

Algorithms for numerical solution of Eq. (1) have been proposed for example by Henrici [4], Squier [6], Babuška et al. [1], Hayashi [3] and Ohashi [5]. These authors analyzed their methods with the function ω in (2) of the form $\omega(t, u) = Lu$, $L \geqq 0$.

The methods considered in the present paper are a slight generalization of those mentioned above. Here we study the general method defined by a sequence $\{v_{N,n}\}_{n=0}^{N}$ such that the expression

$$\|v_{N,n+1} - \Phi_{N,n}(t_{N,n+1})\|$$

is convergent to zero as $N \to \infty$ and $t_{N,n} \in I$ is fixed. In the above expression $\Phi_{N,n}$ denotes a solution of Eq. (1) passing through $(t_{N,n}, v_{N,n})$, $t_{N,n} \in I$.

In Section 2 we study the sequence $\{v_{N,n}\}$ with the above condition. We give there estimations of errors and sufficient conditions under which it is convergent in Dahlquist's sense (see Theorem 1 and Remark 1).

In Section 3 we discuss the one-step methods with the condition $\omega(t, u) = k(t)\, u$, for which the estimations of errors are better than those in [4], [6] and [1].

2. CONVERGENCE

We introduce the following assumptions:

Assumption A. Suppose that

1° the function $f : I \times R^m \to R^m$ is continuous with respect to all variables,

2° there exists a continuous function $\omega : I \times R^1_+ \to R^1_+$ with the properties:

a) $\omega(t, 0) = 0,\ t \in I$,

b) the only solution u of the differential equation $u' = \omega(t, u)$ on the interval I satisfying the initial condition $u(a) = 0$ is $u(t) \equiv 0,\ t \in I$,

3° for $(t, y_i) \in I \times R^m,\ i = 1, 2$ we have

$$\|f(t, y_1) - f(t, y_2)\| \leqq \omega(t, \|y_1 - y_2\|),$$

4° the maximal solution $\mu : (t, t_0, v_0) \to \mu(t, t_0, v_0)$ of the initial problem $z'(t) = \omega(t, z(t)),\ z(t_0) = v_0$ exists for $t \in [t_0, b],\ t_0 \in I,\ v_0 \in R^1$.

Assumption B. We denote the points of a partition of I by $t_{N,0}, t_{N,1}, \ldots, t_{N,N}$ where $a = t_{N,0} < t_{N,1} \ldots < t_{N,N} = b$ and $h_{N,i} = t_{N,i} - t_{N,i-1},\ i \in J'_N$ where $J'_r = \{1, 2, \ldots \ldots, r\}$. Put $\delta_N = \max_{i \in J'_N} h_{N,i}$. We suppose that $\delta_N \to 0$ if $N \to \infty$. Let $\{v_{N,n}\}_{n=0}^N$ be a sequence such that $v_{N,0} = \eta$ and $v_{N,n} \in R^m,\ n \in J'_N$. Let

$$\|v_{N,n+1} - \Phi_{N,n}(t_{N,n+1})\| \leqq b_{N,n+1})\,(h_{N,n+1}),\,,\quad n \in J_{N-1}\,,$$

where $b_{N,n} : [0, h_0] \to R^1_+,\ J_r = \{0, 1, \ldots, r\}$ and $\Phi_{N,n}$ is a solution of Eq. (1) passing through $(t_{N,n}, v_{N,n})$. The element $v_{N,n}$ of the sequence $\{v_{N,n}\}_{n=0}^N$ will be called the approximation of solution of Eq. (1) at the point $t_{N,n}$.

It is known that Assumption A guarantees that there exists a unique continuously differentiable solution y of Eq. (1) (see [2]). We shall assume that this solution is defined for $t \in I$. Here we take the elements of the sequence $\{v_{N,n}\}_{n=0}^N$ for computing the solution y of Eq. (1) at the points $t_{N,n}$. We have the following

Theorem 1. *If Assumptions* A *and* B *are satisfied then we have*

$$\|v_{N,n} - y(t_{N,n})\| \leqq u_{N,n}\,,\quad n \in J_N\,, \tag{3}$$

where

$$\begin{cases} u_{N,0} = 0\,, \\ u_{N,n+1} = \mu(t_{N,n+1}, t_{N,n}, u_{N,n}) + b_{N,n+1}(h_{N,n+1})\,, \quad n \in J_{N-1}\,, \end{cases}$$

and μ *denotes the maximal solution of the initial problem* $z'(t) = \omega(t, z(t))$, $z(t_{N,n}) = u_{N,n}$.

Proof. We shall prove the inequality (3) by induction. It is easy to see that

$$\|v_{N,1} - y(t_{N,1})\| = \|v_{N,1} - \Phi_{N,0}(t_{N,1})\| \leqq b_{N,1}(h_{N,1})\,.$$

Further, we assume that for positive integers $n > 1$ the inequality (3) is true. On the interval $[t_{N,n}, t_{N,n+1}]$ the functions $\Phi_{N,n}$ and y are the solutions of Eq. (1) passing through $(t_{N,n}, v_{N,n})$ and $(t_{N,n}, y(t_{N,n}))$, respectively.

Now we have

$$\|v_{N,n+1} - y(t_{N,n+1})\| \leqq \|v_{N,n+1} - \Phi_{N,n}(t_{N,n+1})\| + \\ + \|\Phi_{N,n}(t_{N,n+1}) - y(t_{N,n+1})\| \leqq u_{N,n+1},$$

and so the theorem is proved by induction.

Remark 1. If

1° $b_{N,i}(h_{N,i}) \leqq b_{N,i+1}(h_{N,i+1})$, $i \in J'_{N-1}$,

2° the function μ is non-decreasin ; with respect to the last variable,

3° $\lim_{N\to\infty} u^*_{N,N} = 0$ where $u^*_{N,0} = 0$, $u^*_{N,n+1} = \mu(b, a, u^*_{N,n}) + b_{N,n+1}(h_{N,n+1})$, $n \in J_{N-1}$

then

(5) $$\|v_{N,n} - y(t_{N,n})\| \to 0 \text{ as } N \to \infty \text{ with } t_{N,n} \text{ fixed, } t_{N,n} \in I.$$

Of course, it is easy to prove by induction that

$$u^*_{N,i} \leqq u^*_{N,i+1}, \quad i \in J_{N-1},$$

and

$$u_{N,i} \leqq u^*_{N,i}, \quad i \in J_N.$$

Hence and by (3) we obtain the assertion (5).

3. Case of the function $\omega(u, v)$ linear in v. Now we shall discuss the above problems in a special case, where the function ω is linear with respect to the last variable, i.e. fulfils the condition

(6) $$\omega(u, v) = k(u)\, v,$$

and the function $k: I \to R^1_+$ is a Lebesgue-integrable function. Let $h_N = (b - a)/N$, i.e. $h_{N,i} = h_N$, $i \in J'_N$ and $t_{N,j} = a + jh_N$, $j \in J_N$.

Definition (see also [6]). *The elements of the sequence* $\{v_{N,n}\}$ *of Assumption* B *are* $b_{N,n}$*-consistent with Eq.* (1) *if*

(7) $$\lim_{N\to\infty} \sum_{i=1}^{N} b_{N,i}(h_N) = 0,$$

where the functions $b_{N,i}$ *are defined in Assumption* B.

Remark 2. If $b_N: [0, h_0] \to R^1_+$, $\lim_{N\to\infty} b_N(h_N) = 0$ and $b_{N,i}(h_N) = h_N\, b_N(h_N)$ then the condition (7) holds.

Theorem 1 implies the following

Theorem 2. *If Assumption* A *is satisfied with the condition* (6) *and the sequence* $\{v_{N,n}\}$ *is* $b_{N,n}$*-consistent with Eq.* (1) *then*

(8) $$\|v_{N,n} - y(t_{N,n})\| \leqq \sum_{i=1}^{n} b_{N,i}(h_N) \exp\left(\int_{t_{N,i}}^{t_{N,n}} k(\tau)\, d\tau\right), \quad n \in J'_N,$$

and

(9) $$\|v_{N,n} - y(t_{N,n})\| \to 0 \text{ as } N \to \infty \text{ with } t_{N,n} \text{ fixed, } t_{N,n} \in I.$$

Proof. By the Gronwall inequality which is an "integrated" analogue to the inequality (4) we have

$$\|\Phi_{N,n}(t) - y(t)\| \leqq u_{N,n} \exp\left(\int_{t_{N,n}}^{t} k(\tau)\,\mathrm{d}\tau\right),$$

i.e.

$$\mu(t, t_{N,n}, u_{N,n}) = u_{N,n} \exp\left(\int_{t_{N,n}}^{t} k(\tau)\,\mathrm{d}\tau\right)$$

and hence we obtain the estimation (8). Since the sequence $\{v_{N,n}\}$ is $b_{N,n}$-consistent with Eq. (1), (8) implies (9). This completes the proof.

The sequence $\{v_{N,n}\}$ may be generated in a variety of ways. If the solution $\Phi_{N,n}$ of Eq. (1) passing through $(t_{N,n}, v_{N,n})$ is $m + 1$ times continuously differentiable, then we obtain from Taylor's formula

$$v_{N,n+1} = \Phi_{N,n}(t_{N,n} + h_N) = v_{N,n} + \sum_{i=1}^{m} \frac{h_N^i}{i!} \Phi_{N,n}^{(i)}(t_{N,n}) + O(h_N^{m+1})$$

for $n \in J_{N-1}$ and $b_{N,i}(h_N) = 0$, $i \in J'_N$.

If we put

$$v_{N,n+1} = v_{N,n} + \sum_{i=1}^{m} \frac{h_N^i}{i!} \Phi_{N,n}^{(i)}(t_{N,n}), \quad n \in J_{N-1}$$

then $b_{N,i} = O(h_N^{m+1})$ and the sequence $v_{N,n}$ is consistent with Eq. (1) for $m > 0$.

Henrici [4], Babuška, Práger and Vitásek [1] and Ohashi [5] considered the sequence $\{v_{N,n}\}$ defined by

$$\begin{cases} v_{N,0} = \eta, \\ v_{N,n+1} = v_{N,n} + h_N P(t_{N,n}, v_{N,n}, h_N) + r_N(h_N), \quad n \in J_{N-1}, \end{cases}$$

while Squier [6] studied the method of the form

$$\begin{cases} \tilde{v}_{N,0} = \eta, \\ \tilde{v}_{N,n+1} = G(t_{N,n}, \tilde{v}_{N,n}, h_N), \quad n \in J_{N-1}, \end{cases}$$

where the functions $P, G : I \times R^m \times [0, h_0] \to R^m$ satisfied the adequate conditions.

Here we shall study the one-step method defined by

$$\begin{cases} v_{N,0}^* = \eta, \\ v_{N,n+1}^* = G_{N,n}(t_{N,n}, v_{N,n}^*, h_N), \quad n \in J_{N-1}, \end{cases}$$

where $G_{N,n} : I \times R^m \times [0, h_0] \to R^m$.

Let

$$k(t) \leqq L, \quad t \in I, \quad L \geqq 0 \quad \text{and} \quad b_{N,i}(h_N) = h_N\, b_N(h_N).$$

In this case, by (8) we have

$$\|v_{N,n}^* - y(t_{N,n})\| \leqq b_N(h_N)\, E_L(t_{N,n} - a), \quad n \in J_N,$$

where

$$E_L(x) = \begin{cases} x & \text{if } L = 0, \\ \dfrac{\exp(Lx) - 1}{L} & \text{if } L \neq 0. \end{cases}$$

Further, if $\lim_{N \to \infty} b_N(h_N) = 0$ then Theorem 2 yields the results contained in [6], [4] and [1]. Since $\int_c^d k(\tau)\,d\tau \leqq L(d - c)$ for $a \leqq c \leqq d \leqq b$, Theorem 2 gives a better result than those known so far.

By Assumption A and $t \in [t_{N,n}, t_{N,n+1}]$ we have

$$(4) \qquad D_R \|\Phi_{N,n}(t) - y(t)\| \leqq \|\Phi'_{N,n}(t) - y'(t)\| \leqq \omega(t, \|\Phi_{N,n}(t) - y(t)\|),$$

where $D_R z$ denotes the right derivative of the function z. Hence and from the theory of differential inequalities (see [2], p. 26) we obtain

$$\|\Phi_{N,n}(t) - y(t)\| \leqq \mu(t, t_{N,n}, v_{N,n}).$$

Let

$$(10) \qquad G_{N,n}(t, x, h) = g(x) + h^\alpha P(t, x, h) + O(h^{\alpha+\beta}), \quad \beta > 0, \ \alpha \in R^1,$$

where $g : R^m \to R^m$, $P : I \times R^m \times [O, h_0] \to R^m$. Moreover, let

$$(11) \qquad \left\| P(t, v, h) - \frac{z(t + h) - g(v)}{h^\alpha} \right\| \leqq d(h), \ t \in I, \ v \in R^m, \ h \in [O, h_0],$$

where $d : [O, h_0] \to R^1_+$ and z is a solution of Eq. (1) on the interval $[t, t + h]$ such that $z(t) = v$. It is easy to see that if $\lim_{N \to \infty} h_N^\alpha\, d(h_N) = 0$ then the sequence $\{\tilde{v}_{N,n}\}$ is $b_{N,n}$-consistent with Eq. (1) with $b_{N,n} = h_N^\alpha\, d(h_N) + O(h_N^\beta)$.

Put $g(x) \equiv x$, $\alpha = 1$, $d(h) = Mh^p$, $M \geqq 0$, $p > 0$. Then the condition (11) indicates that the method defined by the sequence $\{\tilde{v}_{N,n}\}$ (or the function P) is of the order p. In this case all the assumptions of Theorem 2 are satisfied and we get the results of Theorem 3.3 [4].

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Souhrn

POZNÁMKY K NUMERICKÉMU ŘEŠENÍ POČÁTEČNÍ ÚLOHY PRO SOUSTAVY DIFERENCIÁLNÍCH ROVNIC

TADEUSZ JANKOWSKI

V článku je uvedena třída numerických metod pro přibližné řešení soustav obyčejných diferenciálních rovnic. Je dokázáno, že za jistých obecných podmínek tyto metody konvergují pro dostatečně malý krok. Jsou uvedeny odhady chyb, které jsou lepší než dosud známé.

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