van Huu Nguyen

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PARAMETRIC TEST FOR CHANGE IN A PARAMETER OCCURRING IN THE DENSITY OF ONE-PARAMETER EXPONENTIAL FAMILY

NGUYEN-VAN-HUU

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1. INTRODUCTION

Let $X_1, \ldots, X_N$ be independent random variables where $X_i$ has the one-parameter exponential density with respect to a $\sigma$-finite measure $\mu$ of the form:

$$f(x, \theta_i) = h(x) \exp \left( \psi_1(\theta_i) U(x) + \psi_2(\theta_i) \right), \quad i = 1, 2, \ldots, N.\tag{1}$$

Let us consider the problem of testing $H_0$ against a class of alternatives $K = \{K_1, \ldots, K_s\}$ defined by

$$H_0 : \theta_1 = \ldots = \theta_N = \theta_0,$$

with $\theta_0$ known,

$$K_i : \theta_1 = \theta_0 + \Delta C_{i1}; \ldots; \theta_N = \theta_0 + \Delta C_{iN}, \quad i = 1, 2, \ldots, s,$$

where $\Delta$ is unknown, and $C_{ij}$ are so-called regression constants. $K_i$ is called the regression alternative.

A special case of this problem where

$$C_{i1} = \ldots = C_{ii} = 0; \quad C_{i,i+1} = \ldots = C_{iN} = 1$$

for $i = 1, \ldots, N - 1$, has been investigated by Kander and Zacks [2].

2. LOCALLY AVERAGE MOST POWERFUL (LAMP) TEST

**Theorem 1.** Suppose that $\psi_1(\theta)$ is increasing, and $\psi_1(\theta), \psi_2(\theta)$ have finite first order derivatives $\psi_1'(\theta), \psi_2'(\theta)$ on $\Omega$ — the parametric space.
For testing $H_0$ against $\{K_1, \ldots, K_s\}$ let us consider the test defined by the critical function

\[(4) \quad \phi(X) = 1, \gamma, 0 \quad \text{if} \quad T_{Np}(U) >, =, < C_2, \]

where

\[(5) \quad T_{Np}(U) = \sum_{j=1}^{N} C_j(p) U(X_j), \quad C_j(p) = \sum_{m=1}^{s} C_{mj} P_m, \]

and $\gamma, C_2$ are defined so that the test has the level of significance $\alpha$, $p = (p_1, \ldots, p_s)$, $\sum_{m=1}^{s} p_m = 1$, are the weights associated to the alternatives $K_1, \ldots, K_s$. Then there exists an $\epsilon > 0$ such that for all $0 < \Delta \leq \epsilon$, the sum $\sum_{m=1}^{s} P_m E_m \Phi'(X)$ attains the maximum value at $\Phi$ within the class $\{\Phi'\}$ of all possible $\alpha$-level tests where $E_i$ denotes the expectation with respect to $K_i$.

Proof. Put

\[
 f_p(x, \theta_0, \Delta) = \sum_{i=1}^{s} p_i \prod_{j=1}^{N} h(x_j) \exp \left( \sum_{j=1}^{N} \psi_1(\theta_0 + \Delta C_{ij}) U(X_j) + \psi_2(\theta_0 + \Delta C_{ij}) \right); \]

then $f_p(x, \theta_0, 0)$ is the joint density of $X = (X_1, \ldots, X_N)$ under $H_0$. Let $\Phi'(x)$ be any test of $H_0$. We have

\[
(6) \quad \sum_{m=1}^{s} p_m E_m \Phi'(X) = \int \Phi'(x) f_p(x, \theta_0, \Delta) \, d\mu(x), \]

\[
(7) \quad E_0 \Phi'(X) = \int \Phi'(x) f(x, \theta_0) \, d\mu(x), \]

with $f(x, \theta_0) = f_p(x, \theta_0, 0)$. It follows from (6), (7) that the problem of finding the test maximizing the average power $\sum_{m=1}^{s} p_m E_m \Phi'(X)$ within the class of all $\alpha$-level tests reduces to the problem of finding the most powerful test for testing $H_0$ against a simple alternative $f_p(x, \theta_0, \Delta)$ with $\Delta$ fixed. The test, by Neyman-Pearson’s Lemma, is defined by

\[
(8) \quad \phi(X) = 1, \gamma, 0 \quad \text{if} \quad f_p(X, \theta_0, \Delta) >, =, < C_2 f(X, \theta_0) \]

where $\gamma, C_2$ are constants chosen suitably. By some elementary calculation, it is easy to see that

\[
(9) \quad f_p(X, \theta_0, \Delta) f(X, \theta_0) = 1 + \Delta \psi_1(\theta_0) \sum_{j=1}^{N} C_j(p) U(X_j) + \psi_2(\theta_0) \sum_{j=1}^{N} C_j(p) + O(\Delta^2). \]
Since $\psi'_1(\theta_0) > 0$, there exists an $\varepsilon > 0$ such that (8) is equivalent to (4) for each $\theta_0$ fixed and for all $0 < \Delta \leq \varepsilon$.

**Remark.** The test possessing the property defined in Theorem 1 is said to be LAMP. Suppose that the regression constants $C_j$'s take on the form (3); then putting $p_1 = \ldots = p_{N-1} = 1/(N-1)$, we obtain from (4), (5)

\begin{equation}
\phi(X) = 1, \gamma, 0 \quad \text{if} \quad \sum_{j=2}^{N} (j - 1) U(X_j) >, =, < C_x.
\end{equation}

This test was suggested by Kander and Zacks in [2].

The following theorem states that under some restrictions placed on $C_j(p)$ and $U(x)$ the test statistic given by (5) is asymptotically normal.

**Theorem 2.** Assume that $X_1, X_2, \ldots, X_N, \ldots$ are any independent random variables possessing the distribution functions $F_1(x), F_2(x), \ldots, F_N(x), \ldots$, respectively. Further, suppose that $0 < M \leq \text{var} U(X_j) < \infty$ for all $j$, and that $U(x)$ is uniformly square integrable in $F_j(x)$, i.e. for any $\varepsilon > 0$ there exists an $A > 0$ depending only on $\varepsilon$ but not on $j$ such that $\int_{|x| \geq A} U^2(x) \, dF_j(x) < \varepsilon$ uniformly for all $j$. Then the test statistic $T_{N,p}(U)$ given by (5) is asymptotically normal $N(\mu_{cp}, \sigma_{cp})$ where

\begin{equation}
\mu_{cp} = E T_{N,p}(U) = \sum_{j=1}^{N} C_j(p) E U(X_j)
\end{equation}

\begin{equation}
\sigma_{cp}^2 = \text{var} T_{N,p}(U) = \sum_{j=1}^{N} C_j^2(p) \text{var} (U(X_j)),
\end{equation}

provided

\begin{equation}
\sum_{j=1}^{N} C_j^2(p)/\max_{j} C_j^2(p) \to \infty.
\end{equation}

**Proof.** Verifying the proof of Theorem V.1.2 in [5] we realize that the assertion of the theorem remains true under the conditions of Theorem 2.

The case where $\theta_0$ is unknown will be treated in the following examples.

**Example 1.** Suppose that $X_j, j = 1, \ldots, N$, has the normal distribution $N(\theta_j, \sigma_j)$ with $\sigma_j$ known, namely $\sigma_j = 1, \theta_j$ being the unknown mean. Then

\begin{align*}
f(x, \theta_j) &= (2\pi)^{-1/2} \exp \left( -1/2 (x - \theta_j)^2 \right) = \\
&= (2\pi)^{-1/2} \exp \left( -x^2/2 \right) \exp \left( \theta_j x - \theta_j^2/2 \right)
\end{align*}

has the form (1) with $U(x) = x, h(x) = (2\pi)^{-1/2} \exp \left( -x^2/2 \right), \psi_1(\theta) = 0, \psi_2(\theta) = \ldots$
For testing \( H_0 \) against \( \{K_1, \ldots, K_s\} \) we can employ the test statistic

\[
T_{Np}(X) = \sum_{j=1}^{N} C_j(p)(X_j - \theta_0), \quad X = (X_1, \ldots, X_N),
\]

which is equivalent to (5) if \( \theta_0 \) is known. On the contrary, when \( \theta_0 \) is unknown, we can expect that the test defined by the statistic obtained from (14) by replacing \( \theta_0 \) by \( \bar{X} = \sum_{j=1}^{N} X_j/N \) will have some optimality property.

Assume that \( \theta_0 \) is unknown and admits a normal prior distribution \( N(0, \tau) \). Then the density

\[
f_m(x, \theta_0, \Delta) = f_m(x, \theta_1, \ldots, \theta_N) = (2\pi)^{-N/2} \exp \left( -\frac{1}{2} \sum_{j=1}^{N} (x_j - \theta_j)^2 \right) = (2\pi)^{-N/2} \exp \left( -\frac{1}{2} \sum_{j=1}^{N} (x_j - \theta_0 - \Delta C_{mj})^2 \right)
\]

with \( x = (x_1, \ldots, x_N) \) may be considered as the conditional density of \( X_1, \ldots, X_N \) under \( K_m \).

The unconditional joint density of \( X \) under \( K_m \) with respect to the prior distribution \( N(0, \tau) \) of \( \theta_0 \) is given by

\[
f_m(x, \Delta) = \frac{1}{\tau \sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, \theta_0, \Delta) \exp \left( -\frac{\theta_0^2}{2\tau^2} \right) d\theta_0 = C(N, \tau) \exp \left\{ -\frac{1}{2} \sum_{j=1}^{N} [x_j - \bar{x} - \Delta(C_{mj} - \bar{C}_m)]^2 - (N/2)(\bar{x} - \Delta\bar{C}_m)^2/(1 + N\tau^2) \right\}
\]

where \( \bar{C}_m = \sum_{j=1}^{N} C_{mj}/N \), and \( C(N, \tau) \) is the constant depending only on \( N \) and \( \tau \). Note that

\[
f_0(x) = f_m(x, 0) = C(N, \tau) \exp \left\{ -\frac{1}{2} \sum_{j=1}^{N} (x_j - \bar{x}) - \frac{1}{2} N(\bar{x})^2/(1 + N\tau^2) \right\}
\]

is the unconditional density of \( X \) under \( H_0 \).

Let \( E_0, E_m \) be the expectations with respect to \( f_0(x), f_m(x, \Delta) \) and let \( \Phi'(X) \) be any test for testing \( f_0(x) \) against \( f_m(x, \Delta) \), \( m = 1, \ldots, s \), with \( \Delta \) fixed. Then it is easy to see that the test maximizing \( \sum_{m=1}^{s} p_m E_m \Phi'(X) \) within the class of all tests satisfying \( E_0 \Phi'(X) \leq \alpha \) is given by

\[
\Phi(X) = 1, 0 \quad \text{if} \quad \sum_{m=1}^{s} p_m f_m(X, \Delta)f_0(X) > , < C_x'.
\]
Note that \( \sum_{m=1}^{s} p_m f_m/f_0 \) may be expanded in the form:

\[
\sum_{m=1}^{s} p_m f_m(x, \Delta)/f_0(x) = \sum_{m=1}^{s} p_m \exp \{ \Delta \sum_{j=1}^{N} (x_j - \bar{x}) (C_{mj} - \bar{C}_m) + N\Delta \bar{C}_m \bar{x}((1 + N\tau^2) + O(\Delta^2)) \}
\]

\[
= 1 + \Delta \sum_{j=1}^{N} (x_j - \bar{x}) (C_j(p) - \bar{C}(p)) + \frac{N\Delta}{1 + N\tau^2} \bar{C}(p) \bar{x} + O(\Delta^2).
\]

Consequently, when \( \Delta \) is small enough and \( \tau \to \infty \), \( \sum_{m=1}^{s} p_m f_m(x, \Delta)/f_0(x) \) is strictly increasing function of \( T_{np}(x) \) where

\[
T_{np}(x) = \sum_{j=1}^{N} (C_j(p) - \bar{C}(p)) (x_j - \bar{x}) = \sum_{j=1}^{N} (C_j(p) - \bar{C}(p)) x_j,
\]
and (15) is equivalent to

\[
\phi(X) = 1, 0 \text{ if } T_{np}(x) > , < C_x
\]
for all \( \Delta \) small enough. Then \( \phi(X) \) may be regarded as a locally Bayesian solution with respect to the normal prior distribution \( N(0, \tau) \) of \( \theta_0 \) when \( \tau \to \infty \) for the problem of testing \( H_0 \) against \( \{K_1, \ldots, K_s\} \) concerning the mean of a normal distribution.

Remark. If the regression constants \( C_{ij} \)'s take on the form (3) and putting \( p_1 = = \ldots = p_{N-1} = 1/(N-1) \), then (16), (17) reduce to

\[
T_{np}(X) = \sum_{j=2}^{N} (j - 1) (X_j - \bar{X}),
\]

\[
\phi(X) = 1, 0 \text{ if } T_{np}(X) > , < C_x,
\]
which have been considered by Chernoff and Zacks in [1].

Example 2. Suppose that \( X_1, \ldots, X_N \) are independent and \( X_j \) is normally distributed \( N(\mu_j, \theta_j) \) where \( \mu_j \) is known, namely \( \mu_j = 0 \) for all \( j \), and \( \theta_j \) is an unknown parameter.

Consider the problem of testing \( H_0 \) against \( \{K_1, \ldots, K_s\} \) where

\[
H_0 : \theta_1 = \ldots = \theta_N = \theta_0, \quad K_i : \theta_i^2 = \theta_0^2(1 + AC_{i1}), \ldots, \theta_N^2 = \theta_0^2(1 + AC_{in})
\]
for \( i = 1, \ldots, s \).
The density of $X_j$ under $H_0$ and $K_i$'s takes on the form

$$f(x, \theta) = (2\pi \theta^2)^{-1/2} \exp\left(-x^2/2\theta^2\right) = (2\pi)^{-1/2} \exp\left(-x^2/2\theta^2 - \frac{1}{2} \log \theta^2\right)$$

which has the form (1) with $U(x) = x^2$, $\psi_1(\theta) = -1/2\theta$, $\psi_2(\theta) = -\frac{1}{2} \log \theta^2$. Consequently, for testing $H_0$ against $\{K_1, \ldots, K_s\}$ the test given by (4), (5) reduces to

$$\Phi(X) = 1, \gamma, 0 \text{ if } T_{np}(X) = \sum_{j=1}^{N} C_j(p) X_j^2/\theta_0^2 > , = , < C_*$$

provided $\theta_0$ is known.

Let us consider the case where $\theta_0$ is unknown.

Assume that $u = 1/\theta_0^2$ is an exponentially distributed random variable with an unknown parameter $\lambda$, i.e. the density of $u$ is given by: $g_u(u) = \lambda \exp(-\lambda u)$ for $u, \lambda > 0$. Thus the function

$$f_i(x, u, A) = (2\pi)^{-N/2} \prod_{j=1}^{N} (1 + AC_{ij})^{-1/2} \exp\left[-(u/2)\sum_{j=1}^{N} x_j^2/(1 + AC_{ij})\right]$$

may be regarded as the conditional density of $X_1, \ldots, X_N$ under $K_i$ when $u$ is given.

The unconditional density of $X_1, \ldots, X_N$ under $K_i$ is defined by

$$f_i(x, u, A) = \lambda \int_0^\infty f_i(x, u, A) \exp\left(-\lambda u\right) du =$$

$$= C_N(\lambda) \prod_{j=1}^{N} (1 + AC_{ij})^{-1/2} \left[2\lambda + \sum_{j=1}^{N} x_j^2/(1 + AC_{ij})\right]^{-(N/2)-1}$$

where $C_N(\lambda)$ is the constant depending only on $N, \lambda$. Note that $f_0(x) = f_i(x, 0) = C_N(\lambda) \left[2\lambda + \sum_{j=1}^{N} x_j^2\right]^{-(N/2)-1}$ is the unconditional density of $X_1, \ldots, X_N$ under $H_0$.

Let $\Phi(X)$ be any test of the hypothesis $f_0(x)$ against the alternatives $\{f_i(x, A), \ldots, f_s(x, A)\}$ with $A$ fixed. Let $E_0, E_i, i = 1, \ldots, s$, be the expectations with respect to the densities $f_0(x)$ and $f_i(x, A)$. Then the test, which maximizes $\sum_{i=1}^{s} p_i E_i \Phi(X)$ within the class of all $\alpha$-level tests is defined by

$$\Phi(X) = 1, \gamma, 0 \text{ if } \sum_{i=1}^{s} p_i f_i(x, A)/f_0(x) > , = , < C_*$$

By some elementary calculation we easily obtain:

$$\sum_{i=1}^{s} p_i f_i(x, A)/f_0(x) =$$

$$= 1 - (N/2) \bar{C}(p) + A(1 + N/2) \sum_{j=1}^{N} C_j(p) X_j^2/(2\lambda + \sum_{j=1}^{N} X_j^2) + O(A^2).$$
If \( A > 0 \), then \( \sum_{i=1}^{s} p_i f_i(X, A) f_0(X) \) is a strictly increasing function of \( \sum_{j=1}^{N} C_j(p) X_j^2 / \sum_{j=1}^{N} X_j^2 \), or, equivalently, of

\[
T_{np}^w(X) = N^{-1} \sum_{j=1}^{N} (C_j(p) - \bar{C}(p)) X_j^2 / S^2
\]

where \( S^2 = N^{-1} \sum_{j=1}^{N} X_j^2 \), and (22) is equivalent to

\[
\phi(X) = 1, \gamma, 0 \quad \text{if} \quad T_{np}^w(X) > \gamma, = \gamma, < C_x
\]

for all \( 0 < A \) small enough.

Thus the test defined by (24) is a locally Bayesian solution of the problem of testing \( H_0 \) against \( \{K_1, \ldots, K_s\} \) with respect to the exponential prior distribution of \( u = 1/\theta_0^2 \) with the parameter \( \lambda \), which has been supposed to tend to zero.

Remark. The distributions of \( T_{np}^w(X) \) given by (16) and \( T_{np}^w(X) \) given by (23) do not depend on \( \theta_0 \).

3. THE ASYMPTOTIC RELATIVE EFFICIENCY

The definition of the asymptotic relative efficiency was given in [3].

Let us now consider the asymptotic relative efficiency of the rank tests considered in [3] with respect to the parametric tests given by (17), (24) for testing hypotheses on the mean and on the variance of a normal distribution.

We say that an \( \alpha \)-level test is based on the test statistic \( T \) if its critical region takes on the form \( \{T > C_x\} \).

Example 1. Let \( X_1, \ldots, X_N \) be independent random variables possessing the normal distributions \( N(\theta_1, 1), \ldots, N(\theta_N, 1) \), respectively. Consider the problem of testing \( H_0 \) against \( \{K_1, \ldots, K_s\} \) defined by (2) with \( \theta_1, \ldots, \theta_N \) being the means of the normal distributions. For testing \( H_0 \) against \( \{K_1, \ldots, K_s\} \) we can employ the parametric test based on \( T_{np}^w(X) \) given by (16) and the rank test based on the rank test statistic \( T_{np}^{w1}(R) \) given by

\[
T_{np}^{w1}(R) = \sum_{j=1}^{N} [C_j(p) - \bar{C}(p)] a^{(1)}_N(R_j)
\]

where \( a^{(1)}_N(j) = EV^{(j)} = E \phi^{-1}(U^{(j)}) \) with \( V^{(1)} < \ldots < V^{(N)} \), \( U^{(1)} < \ldots < U^{(N)} \) being the ordered samples from the standardized normal and from the uniform distribution, respectively. This test was obtained from Corollary 1 in [3].

Assume that the condition

\[
\sum_{j=1}^{N} [C_j(p) - \bar{C}(p)]^2 / \max_j [C_j(p) - \bar{C}(p)]^2 \rightarrow \infty
\]
is fulfilled. Consider the alternative $K$ defined by

$$K : \theta_1 = d_1, \ldots, \theta_N = d_N,$$

and assume that

$$\sum_{j=1}^{N} (d_j - \bar{d})^2 \to b^2 > 0, \quad \max_{j} (d_j - \bar{d})^2 \to 0$$

hold. We shall show that the asymptotic relative efficiency of the test based on $T_{Np}(R)$ with respect to the test based on $T_{Np}(X)$, say $e[T_{Np}(R) : T_{Np}(X)]$, is equal to 1.

As a matter of fact, $T_{Np}(X)$ is normally distributed $N(0, \sigma_{cp})$ under $H_0$, and $N(b_1, \sigma_{cp})$ under $K$, where

$$b_1 = \sum_{j=1}^{N} \left[ C_j(p) - \bar{C}(p) \right] (d_j - \bar{d}),$$

$$\sigma_{cp}^2 = \sum_{j=1}^{N} \left[ C_j(p) - \bar{C}(p) \right]^2;$$

hence the asymptotic power of the test based on $T_{Np}(X)$ is equal to

(25) $$1 - \phi(k_{1-x} - b_1/\sigma_{cp})$$

where $k_{1-x}$ is the 100(1 - $z$) percentage point of the standardized normal distribution function $\phi(x)$. On the other hand, by Theorem 5 and Remark 1 in [3], $T_{Np}^{(1)}(R)$ has the same asymptotic distribution as $T_{Np}(X)$ both under $H_0$ and under $K$; hence the asymptotic power of the test based on $T_{Np}^{(1)}(R)$ is also given by (25), and, by the definition of the asymptotic relative efficiency, $e[T_{Np}(R) : T_{Np}(X)] = 1$.

Example 2. Let $X_1, \ldots, X_N$ be independent random variables, which are normally distributed $N(0, \theta_1), \ldots, N(0, \theta_N)$, respectively. For testing $H_0$ against $\{K_1, \ldots, K_s\}$ defined by (20) with $\theta_0$ unknown we may employ the parametric test based on the test statistic $T_{Np}(X)$ given by (23) and the test based on the rank test statistic

$$T_{Np}^{(2)}(R) = \sum_{j=1}^{N} \left[ C_j(p) - \bar{C}(p) \right] a^{(2)}_N(R_j)$$

where $a^{(2)}_N(j) = E[V^{(j)}]^2 - 1 = E[\phi^{-1}(U^{(j)})]^2 - 1$ with $V^{(j)}$, $U^{(j)}$ being the same as in Example 1. This rank test was obtained from Corollary 2 in [3]. Let us now calculate the asymptotic relative efficiency of the test based on $T_{Np}^{(2)}(R)$ with respect to the test based on $T_{Np}(X)$ under the alternative $K$ defined by

$$K : \theta_1 = \theta_0^2(1 + d_1), \ldots, \theta_N = \theta_0^2(1 + d_N)$$

with $1 + d_j \geq \delta > 0$ for all $j$. 

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Suppose that $\sum_{j=1}^{N} \frac{(C_j(p) - \bar{C}(p))^2}{\max_j (C_j(p) - \bar{C}(p))^2} \to \infty$, and that

$$\sum_{j=1}^{N} (d_j - \bar{d}_N)^2 / (1 + \bar{d}_N)^2 \to b_s^2 > 0, \quad \max_j (d_j - \bar{d}_N)^2 / (1 + \bar{d}_N)^2 \to 0$$

with $\bar{d}_N = N^{-1} \sum_{j=1}^{N} d_j$.

Without loss of generality we may assume that $\theta_0 = 1$ since the distributions of $T_{Np}^{(2)}(R)$ and $T_{Np}^{(2)}(X)$ do not depend on $\theta_0$ under $H_0$ and $K$. Under these assumptions we shall show that the asymptotic relative efficiency of the test based on $T_{Np}^{(2)}(R)$ with respect to the test based on $T_{Np}^{(2)}(X)$ is equal to 1.

As a matter of fact, by Theorem 5 and Remark 3 in [3], the test statistic $T_{Np}^{(2)}(R)$ is asymptotically normal $N(0, \sigma_{cp})$ under $H_0$, and $N(b_2 / (1 + \bar{d}_N), \sqrt{2} \sigma_{cp})$ under $K$ where

$$b_2 = \sum_{j=1}^{N} (C_j(p) - \bar{C}(p))(d_j - \bar{d}_N), \quad \sigma_{cp}^2 = \sum_{j=1}^{N} (C_j(p) - \bar{C}(p))^2;$$

hence the asymptotic power of the test based on $T_{Np}^{(2)}(R)$ is equal to

$$(26) \quad 1 - \phi(k_{1-x} - b_2 / \sigma_{cp} \sqrt{2} (1 + \bar{d}_N)).$$

We shall now show that the test statistic $T_{Np}^{(2)}(X)$ is asymptotically normal both under $H_0$ and under $K$ with the same asymptotic mean and variance as $T_{Np}^{(2)}(R)$.

Actually, first assume that $\bar{d}_N = \bar{d} = N^{-1} \sum_{j=1}^{N} d_j \to d_0 > -1$. Then $S^2$ converges with probability 1 to 1 under $H_0$, and to $1 + d_0$ under $K$. On the other hand, $N^{-1} \sum_{j=1}^{N} (C_j(p) - \bar{C}(p)) X_j^2$ is, by Theorem 2 (the condition on uniform square integrability of $U(x) = x^2$ in the normal distribution functions $N(0, 1 + d_j)$ is satisfied by the assumption that $\bar{d}_N$ is bounded and $\max (d_j - \bar{d}_N) \to 0)$, asymptotically normal $N(0, \sqrt{2} \sigma_{cp}/N)$ under $H_0$, and $N(b_2/N, \sqrt{2} \sigma_{cp}/N)$ under $K$, where

$$\sigma_{cp}^2 = \sum_{j=1}^{N} (C_j(p) - \bar{C}(p))^2 [1 + d_j]^2 =$$

$$= \sum_{j=1}^{N} (C_j(p) - \bar{C}(p))^2 [1 + \bar{d}_N]^2 [1 + 0(d_j - \bar{d}_N)] \sim$$

$$\sim \sum_{j=1}^{N} (C_j(p) - \bar{C}(p))^2 (1 + \bar{d}_N)^2 \sim (1 + d_0)^2 \sigma_{cp}^2$$

since $\max (d_j - \bar{d}_N)^2 \to 0$. Consequently, by Proposition X, Chapter II, in [4], $T_{Np}^{(2)}(X)$ is asymptotically normal $N(0, \sqrt{2} \sigma_{cp}/N)$ under $H_0$, and $N(b_2/N(1 + \bar{d}_N), \sqrt{2} \sigma_{cp}/N)$ under $K$. 

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Further, the assertion about the asymptotic normality of $T_{N_p}^*(X)$ under $K$ remains true if we only assume that $d_N$ is bounded.

As a matter of fact, assume, on the contrary, that $T_{N_p}^*(X)$ is not asymptotically normal $N(b_2/N(1 + d_N), \sqrt{(2)} \sigma_{cp}/N)$ under $K$. Then there exists a sequence $\{N_v\}$ such that this assumption holds for every subsequence of $\{N_v\}$. Thus passing to a proper subsequence, if necessary, we may assume, without loss of generality, that $N_v \to \infty$ and $d_{N_v} \to d_0$ since $d_N$ is bounded. By the above argument, $T_{N_p}^*(X)$ is asymptotically normal $N(b_2/N_v(1 + d_{N_v}), \sqrt{(2)} \sigma_{cp}/N_v)$ and this contradicts the above assumption. Finally, it follows that the asymptotic relative efficiency of the test based on $T_{N_p}^*(R)$ with respect to the test based on $T_{N_p}^*(X)$ is equal to 1.

References


Souhrn

PARAMETRICKÝ TEST PRO ZMĚNU PARAMETRU
V HUSTOTĚ JEDNOPARAMETRICKÉ EXPORENČIÁLNÍ RODINY

NGUYEN-VAN-HUU

Vyšetřuje se problém testování hypotézy, že pozorování jsou nezávislá identicky rozložená, proti třídě alternativ regrese v parametru, a to pro jednoparametrickou exponenciální rodinu. Odvozuje se parametrický test pro tento problém a rovněž jeho relativní eficience vzhledem k pořadovému testu navrženému autorem v předcházející publikaci.

Author’s address: Dr. Nguyen-van-Huu, Can bo giang day Khoa Toan, truong Dai hoc Tong Hop, Ha-Noi, Viet-Nam.