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SIMULTANEOUS RANK TEST PROCEDURES

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1. INTRODUCTION

Let \( X_j = (X_{1j}, \ldots, X_{pj})' \), \( j = 1, \ldots, N \), be independent \( p \)-dimensional random variables with continuous distribution functions. Consider the hypotheses of randomness associated with some marginal distributions:

\[ H_v : F_j(x) = F^v(x^v), \quad j = 1, \ldots, N, \quad v = 1, \ldots, r, \]

where \( F_j(x) \) is the marginal distribution of the subvector \( X^v, v = 1, \ldots, r, x^1, \ldots, x^r \) is a partition of the vector \( x \), i.e., \( x = (x^1', \ldots, x^r')' \). We are interested in testing hypotheses \( H_1, \ldots, H_r \) and \( H_0 = \bigcap_{v=1}^{r} H_v \) against alternatives \( A_1, \ldots, A_r \) and \( A_0 = \bigcup_{v=1}^{r} A_v \), resp., where \( A_v : F_j(x) = F^v(x^v; \theta_j^v), j = 1, \ldots, N, v = 1, \ldots, r, \) with \( \theta_j = (\theta_j^1, \ldots, \theta_j^r)' \) being a vector of unknown parameters.


Here we give three test procedures analogous to those proposed by Krishnaiah in [5–6] and based on the asymptotic distributions of quadratic rank statistics (for definition see (3) below).

Put

\[ S_c = (S_{c1}, \ldots, S_{cp})', \]

\[ S_{ci} = N^{-1} \sum_{j=1}^{N} (c_{ij} - \bar{c}_i) a_{Nj}(R_{ij}), \quad i = 1, \ldots, p, \]

with \( R_{ij} \) being the rank of \( X_{ij} \) in the sequence \( X_{i1}, \ldots, X_{iN}, c_{ij} \) regression constants, \( a_{Nj}(j) \) scores and \( \bar{c}_i = N^{-1} \sum_{j=1}^{N} c_{ij} \). Denote by \( S_{cv} \) the subvector of \( S_c \) corresponding to \( X^v, v = 1, \ldots, r \). Define

\[ Q_c = S_c' (\text{var}_p S_c)^{-1} S_c, \]

\[ Q_{cv} = S_{cv}' (\text{var}_p S_{cv})^{-1} S_{cv}, \quad v = 1, \ldots, r, \]

\[ 33 \]
where the matrix $\text{var}_p \mathbf{S}_c$ is regular with elements

$$(N - 1)^{-1} \sum_{j=1}^{N} (c_{ij} - \bar{c}_i) (c_{ij} - \bar{c}_i) \sum_{m=1}^{N} (a_{N_i} (R_{im}) - \bar{a}_{N_i}) (a_{N_i} (R_{ij}) - \bar{a}_{N_i})$$

if

$$i, t \in I_k, \quad k = 1, \ldots, r,$$

and

$$\sum_{j=1}^{N} (c_{ij} - \bar{c}_i) (c_{ij} - \bar{c}_i) (a_{N_i} (R_{ij}) - \bar{a}_{N_i}) (a_{N_i} (R_{ij}) - \bar{a}_{N_i})$$

if

$$i \in I_k, \quad t \notin I_k, \quad k = 1, \ldots, r,$$

where $I_1, \ldots, I_r$ is the partition of the set $I = \{1, \ldots, p\}$ considered in hypotheses $H_v$ and $\text{var}_p \mathbf{S}_c^v$ is the submatrix of $\text{var}_p \mathbf{S}_c$ corresponding to $\mathbf{S}_c^v$ and $\bar{a}_{N_i} = N^{-1} \sum_{j=1}^{N} a_{N_i}(j)$.

Denote by $m_v$ the number of components of $\mathbf{x}^v, \nu = 1, \ldots, r$.

We shall impose usual conditions on scores, regression constants and the matrix $\text{var}_p \mathbf{S}_c$:

a. The scores $a_{N_i}(j)$ are generated by a nonconstant square integrable functions $\varphi_i, \ i = 1, \ldots, p, \ i.e.,$

$$\int_{0}^{1} (\varphi_i(u) - a_{N_i}([uN] + 1))^2 \, du \to 0 \quad \text{for} \quad N \to \infty, \ i = 1, \ldots, p.$$

b. The regression constants fulfill:

$$(5) \quad \max_{1 \leq j \leq N} (c_{ij} - \bar{c}_i)^2 \left( \sum_{j=1}^{N} (c_{ij} - \bar{c}_i)^2 \right)^{-1} \to 0, \quad i = 1, \ldots, p.$$

c. The matrices $\text{var}_p \mathbf{S}_c$ are regular and any accumulation point of the set $\{E \text{var}_p \mathbf{S}_c; \ c_{ij}'s \ satisfy \ (5)\}$ is a regular matrix.

In the sequel we shall often use the following results:

A. Under hypothesis $H_0$ and assumptions a, b, c the asymptotic distribution of $\mathbf{S}_c$ is multivariate normal $\mathcal{N}(0, \text{var} \mathbf{S}_c)$, where $\text{var} \mathbf{S}_c$ is the variance matrix of $\mathbf{S}_c$ under hypothesis $H_0$ (see [2]).

B. Under hypothesis $H_0$ and assumptions a, b, c the asymptotic distributions of $Q_c$ and $Q_c^1, \ldots, Q_c^r$ are $\chi^2$ with $p$ and $m_1, \ldots, m_r$ degrees of freedom, resp. (see [2]).

C. Under hypothesis $H_0$ and assumptions a, b, c the matrix $\mathbf{S}_c \mathbf{S}_c^t$ has asymptotically central Wishart distribution with 1 degree of freedom and positive definite matrix $\text{var} \mathbf{S}_c$ (it follows from A).

D. Under hypothesis $H_0$ and assumptions a, b, c the joint asymptotic distribution of $Q_c^1, \ldots, Q_c^r$ is the generalized multivariate $\chi^2$-distribution defined by Jensen in [4], where the corresponding density is derived (it follows from C and [4]).

E. For an arbitrary subvector $\mathbf{S}_c^v$ of $\mathbf{S}_c$ the relation

$$\mathbf{S}_c^v (\text{var}_p \mathbf{S}_c^v)^{-1} \mathbf{S}_c^v = \max_{u \neq 0} \frac{(u \mathbf{S}_c^v)^2}{u^t \text{var}_p \mathbf{S}_c^v u},$$

where the matrix $\text{var}_p \mathbf{S}_c$ is regular with elements
where \( \mathbf{u} \) are nonzero real vectors, holds and thus
\[
S_c^*(\text{var}_p, S_c^*)^{-1} S_c^* \leq Q_c
\]
(as follows by Schwarz inequality).

F. Bonferroni inequality: For arbitrary events \( A_1, \ldots, A_r \) the inequality
\[
P(\bigcap_{i=1}^r A_i) \geq 1 - \sum_{i=1}^r (1 - P(A_i))
\]
is true.

G. Let a random \( p \)-vector \( \mathbf{Y} = (Y_1, \ldots, Y_p)' = (\mathbf{Y}', \ldots, \mathbf{Y}')' \) have the normal distribution \( \mathcal{N}(\mathbf{0}, \Sigma) \), where
\[
\Sigma = \begin{pmatrix}
\Sigma_{11}, & \ldots, & \Sigma_{1r} \\
\vdots & & \vdots \\
\Sigma_{r1}, & \ldots, & \Sigma_{rr}
\end{pmatrix}.
\]

Assume that there exist vectors \( \mathbf{b}_i \) with \( m_i \) components, \( i = 1, \ldots, r \), \( \sum_{i=1}^r m_i = p \), such that
\[
(6) \quad \Sigma_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j', \quad i \neq j, \ i, j = 1, \ldots, r,
\]
\[
(7) \quad \Sigma_{ii} - \mathbf{b}_i \cdot \mathbf{b}_i' \geq 0, \quad i = 1, \ldots, r,
\]
then for arbitrary convex sets \( C_1, \ldots, C_r \) symmetric about origin, \( C_i \subset \mathbb{R}_{m_i} \), the inequality
\[
P(\mathbf{Y}^i \in C_i, \ i = 1, \ldots, r) \geq \prod_{i=1}^r P(\mathbf{Y}^i \in C_i)
\]
holds (see [1]).

The inequality always holds for \( m_i = 1, \ i = 1, \ldots, r \) (see [10]).

2. TEST PROCEDURES

Procedure I. The author [2] proposed the test procedure with critical regions
\[
(8) \quad Q_c > \chi^2_1(p),
\]
where \( \chi^2_1(p) \) is 100\( \alpha \)% critical value of the central \( \chi^2 \)-distribution with \( p \) degrees of freedom. This test can be used for a class of hypotheses that contain \( H_0 \) as a sub-hypothesis, e.g. for hypothesis that all \( \mathbf{X}_j, \ j = 1, \ldots, N \), have the same distributions.

Procedure II. We base the test procedure on the statistics \( Q_c^1, \ldots, Q_c^r \) given by (4).

We reject the hypothesis \( H_v \) if
\[
Q_c^v > d_v
\]
where the \( d_v \)'s are chosen so that
\[
\lim_{c} P(Q_c^v < d_v, \ v = 1, \ldots, r) = 1 - \alpha.
\]
The total hypothesis \( H_0 \) is rejected if at least one of the hypotheses \( H_1, \ldots, H_r \) is
rejected. The optimal choice of the $d'_v$s is not known. Consistently with the classical normal case the values $d_1, \ldots, d_r$ are chosen either to be equal (i.e. $d_1 = \ldots = d_r = d$) or the individual critical regions are of equal sizes (denote them by $d'_1, \ldots, d'_r$). When $m_v = m$, $v = 1, \ldots, r$ then $d'_v = d_v$, $v = 1, \ldots, r$. To find $d, d'_1, \ldots, d'_r$ with the requested properties is also very difficult for the asymptotic distribution of $(Q_1, \ldots, Q_r)$ includes numerous parameters. This problem was discussed by Jensen in [4] where some approximations are suggested.

We shall suggest here three approximations of $d, d'_1, \ldots, d'_r$. First consider the approximation of $d$. Using Bonferroni inequality we get an approximative value $\chi^2_{2, r}(\max m_i)$ and the critical region for testing $H_v$ against $A_v$

\begin{equation}
Q^*_c > \chi^2_{2, r}(\max m_i) .
\end{equation}

When the assumptions in G are satisfied then the critical region is

\begin{equation}
Q^*_c > \chi^2_{1-(1-\alpha)^{1/r}}(\max m_i) .
\end{equation}

Utilizing assertion E we get the third possible approximation of $d$. Then we reject the hypothesis $H_v$ if

\begin{equation}
Q^*_c > \chi^2_2(p) .
\end{equation}

Similarly we obtain the approximations of $d'_1, \ldots, d'_r$. By Bonferroni inequality and by G (if possible) we have the critical regions for testing $H_v$ against $A_v$

\begin{equation}
Q^*_c > \chi^2_{2, r}(m_v) \end{equation}

and

\begin{equation}
Q^*_c > \chi^2_{1-(1-\alpha)^{1/r}}(m_v),
\end{equation}

respectively.

If $m_i = 1, i = 1, \ldots, p$, the test procedure can be based on the statistics $S_{c1}, \ldots, S_{cp}$. Similarly, in the general case we get critical regions

\begin{equation}
|S_{cil}| > \left( \sum_{j=1}^N (e_{ij} - \bar{e}_i)^2 (N - 1) \right)^{-1} \sum_{v=1}^N (a_N(v) - \bar{a}_N)^2 \left( 1 - \frac{\alpha}{2} \right),
\end{equation}

\begin{equation}
|S_{cil}| > \left( \sum_{j=1}^N (e_{ij} - \bar{e}_i)^2 (N - 1) \right)^{-1} \sum_{v=1}^N (a_N(v) - \bar{a}_N)^2 \left( \frac{1}{2} (1 - \alpha)^{1/p} \right),
\end{equation}

\begin{equation}
|S_{cil}| > \left( \sum_{j=1}^N (e_{ij} - \bar{e}_i)^2 (N - 1) \right)^{-1} \sum_{v=1}^N (a_N(v) - \bar{a}_N)^2 \left( \chi^2_2(p) \right)^{1/2},
\end{equation}

where $u(\cdot)$ is the $100\alpha$% quantile of the normal distribution (0, 1).

As for the comparison of the critical regions (9—10), (12—13), we can easily get the following relations among the approximations of $d_1, \ldots, d_r$

\begin{align*}
\chi^2_{1-(1-\alpha)^{1/r}}(\max m_i) & \geq \chi^2_{1-(1-\alpha)^{1/r}}(m_v), \\
\chi^2_{2, r}(\max m_i) & \geq \chi^2_{2, r}(m_v) \geq \chi^2_{1-(1-\alpha)^{1/r}}(m_v), & v = 1, \ldots, r .
\end{align*}
Thus the critical region (13) is larger than (9), (10) and (12). The comparison of (11) with the other critical regions is more complicated, e.g.

- If $\alpha \leq 0.05$, $p - \max m_i \geq 5$ then $\chi^2_{\alpha/\max m_i}(p)$.
- If $\alpha = 0.05$, $p = 22$, $\max m_i \leq p - 2$ then $\chi^2_{0.05}(p) < \chi^2_{1-(0.95)^{1/\max m_i}}(p)$.

When $m_i = 1$ then the largest critical region is (15).

**Procedure III.** Define

$$Q_{cv}^* = S^*(\var_p S^*)^{-1} S^*,$$

where

$$S^*_{ct} = S^*_c,$$

$$S^*_{cv+1} = S^*_{cv+1} - \cov_p (S^*_{v+1}; S^*_c),...S^*_c) (\var_p (S^*_c,...,S^*_r))^{-1},$$

$$\cov (S^*_{v+1}; S^*_c,...,S^*_r) = (\cov_p (S^*_{v+1}; S^*_c),...\cov_p (S^*_{v+1}; S^*_r)).$$

$$\var_p S^*_{cv+1} = \var_p S^*_{v+1} - \cov_p (S^*_{v+1}; S^*_c,...,S^*_r) (\var_p (S^*_c,...,S^*_r))^{-1},$$

and $Q_{,cv}^*$ has asymptotically $\chi^2$-distribution with $m_i$ degrees of freedom. By direct computations we get that $S^*_1,...,S^*_r$ are asymptotically independent and thus so are $Q_{c1},...,Q_{cr}$.

Using these arguments one can assert that

$$\lim P(Q_{cr}^* < \chi^2_{1-(1-\alpha)^{1/\max m_i}}, v = 1,...,r) \geq$$

$$\lim P(Q_{cr}^* < \chi^2_{1-(1-\alpha)^{1/(m_i)}}, v = 1,...,r) = 1 - \alpha .$$

Thus the critical region for testing the hypothesis $H_v$ against $A_v$ can be chosen in either of the following ways:

1. $Q_v^* > \chi^2_{1-(1-\alpha)^{1/\max m_i}},$  
2. $Q_v^* > \chi^2_{1-(1-\alpha)^{1/(m_v)}}.$

Obviously, the critical region (18) contains (17).

We reject the hypothesis $H_0$ if we reject at least one of $H_1,...,H_r$.
If \( m_i = 1, i = 1, \ldots, p \) the test procedure can be based on the statistics \( S^*_v \), \( v = 1, \ldots, p \). We reject the hypothesis \( H_v \) if

\[
|S^*_v| > (\text{var} S^*_v)^{1/2} \left( \frac{1}{2} + \frac{1}{2}(1 - \alpha)^{1/p} \right).
\]

References


Souhrn

MARIE HOŠKOVÁ

SIMULTÁNNÍ PROCEDURY POŘADOVÝCH TESTŮ

Nechť \( X_j, j = 1, \ldots, N \) jsou nezávislé \( p \)-rozměrné náhodné vektory se spojitou distribuční funkcí \( F_j \). V článku jsou navržena tři testová kritéria založená na pořadích pro test nezávislosti marginálních rozdělení \( X_j \) na indexu \( j \). Výchozím bodem pro konstrukci testových kritérií byl článek P. R. Krishnaiah a (Ann. Inst. Statist. Math. 17, 35–53, 1965).

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