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TESTING OF CONVEX POLYHEDRON VISIBILITY BY MEANS OF GRAPHS

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This paper follows the article [1] which solves the problem of finding the boundary of a convex polyhedron in both parallel and central projections. Our aim is to give a method which yields a simple algorithm for the automation of an arbitrary graphic projection of a convex polyhedron.

To this aim we employ some concepts and features of the graph theory and their interpretation in terms of Boolean matrices.

In the first section of this paper we recall some necessary concepts from the graph theory. The second section applies graphs in order to determine visibility of a convex polyhedron. The term "visibility" is understood in the current sense as used in descriptive geometry.

1. SOME CONCEPTS AND FEATURES OF GRAPH THEORY

An unoriented graph with vertices from a set X and with edges from a set U will be denoted by $\mathcal{G} = (X, U, R)$; R is here a relation defined on ordered triplets of elements x, u, y $(x, y, \in X$ and $u \in U)$ so that $R(x, u, y) = R(y, u, x)$.

Let $X = \{1, 2, \ldots, n\}$ be the set of vertices of a graph $\mathcal{G} = (X, U, R)$ and let $U = \{u_1, u_2, \ldots, u_m\}$ be the set of its edges.

A finite sequence

$$i_0, u_1, i_1, u_2, i_2, \ldots, i_{s-1}, u_s$$

$s \geqq 0$, of elements of a graph $\mathcal{G}$ such that

$$R(i_0, u_1, i_1) \wedge R(i_1, u_2, i_2) \wedge \ldots \wedge R(i_{s-1}, u_s, i_s)$$

is a walk from i_0 to i_s and the number s is the length of this walk.

The distance $\varrho(i, j)$ between two vertices i and j of the graph $\mathcal{G} = (X, U, R)$ is the length of their shortest walk. The number $d(\mathcal{G}) := \max_{i,j \in X} \varrho(i, j)$ is the diameter of the graph $\mathcal{G}$.

Note 1. It is obvious that the diameter $d(\mathcal{G})$ of a graph $\mathcal{G}$ fulfils $d(\mathcal{G}) \leqq n - 1$, where $n = \text{card}\{X\}$.

The matrix of order $n = \text{card}\{X\}$ with elements $g_{ij} \in \{0, 1\}$, $i, j \in \{1, 2, \ldots, n\}$ where $g_{ij} = 1$ if i is adjacent with j (i.e. the vertices i and j are in relation R) and $g_{ij} = 0$ otherwise is called the adjacency matrix G of the graph $\mathcal{G}$.

In the next part of this paper we shall employ Boolean matrices of a graph only, i.e. matrices over Boolean algebra $B\{0, 1\}$ (nevertheless we keep the usual symbols for sum, product and power of a matrix).

We denote $S^k = E + G + G^2 + \ldots + G^k$, where G is a Boolean matrix of the graph $\mathcal{G}$, k is either a natural number or 0 and E is the unit matrix with the same order as the matrix G.

It is obvious that

$$S^0 = E ,$$
$$S^k = (E + G)^k = S^{k-1} + G^k , \quad k > 0 .$$

The elements 1 of the matrix S^k express that between the corresponding vertices of the graph there exists a walk whose length is less or equal to k. The equation $S^0 = E$ means that each vertex is connected with itself by a walk with the length 0.

Note 2. For a computer it is convenient to construct the matrix S^k by

$$S^k = [S^{k-1} . G] + E , \quad k \geqq 1 .$$

It is not difficult to prove this identity.

For Boolean matrices $A \leqq B$ is equivalent to $a_{ij} \leqq b_{ij}$ for all elements of both matrices; so we can write $S^k \leqq S^r$ for $k < r$.

Theorem 1. *For a finite graph $\mathcal{G}$ there always exists a number d, $0 \leqq d \leqq \text{card}\{U\}$ such that*

$$S^d = S^{d+1} = S^{d+2} = \ldots .$$

The number d is the diameter of the graph $\mathcal{G}$.

Proof. From the definition of the graph diameter it follows that the matrix $G^{d(g)}$ included in the matrix S^d describes a walk the exact length of which is $d(\mathcal{G})$ $(d(\mathcal{G}) > 0)$. For $d(\mathcal{G}) = 0$ it is $G^0 = E$. As the walks with lengths $s > d(\mathcal{G})$ are constructed from walks with lengths less or equal to the diameter of the graph $\mathcal{G}$, we find $d(\mathcal{G}) = d$.

Besides $d(\mathcal{G}) \leqq n - 1$, $n = \text{card}\{X\}$ and therefore $d \leqq n - 1$. The matrix S^d can be constructed by a finite number of steps.

Definition 1. *The matrix S^d from Theorem 1 will be called the stable matrix of the graph $\mathcal{G}$.*

Note 3. The construction of the matrix G implies symmetry of this matrix. So both the powers of G and the matrix S^d are symmetrical as well.

Now we shall construct a certain decomposition of the graph $\mathcal{G} = (X, U, R)$ into disjoint subgraphs. Vertices x, y of the graph $\mathcal{G}$ are separated if there exists no walk that connects xy and they are unseparated if there exists at least one walk that connects them.

In accordance with the criterion of "separating" which is an equivalence on the set of vertices X of a graph $\mathcal{G}$ we can decompose the set X into classes $X_1, X_2, \ldots, X_\varkappa$, $\varkappa \geqq 1$, in such a way that each two vertices are unseparated. Then the subgraphs $\mathcal{G}_i = (X_i, U_i, R)$ generated by the sets X_i, $i \in \{1, 2, \ldots, \varkappa\}$, have neither common vertices nor edges and we call them components (of connectivity) of the graph $\mathcal{G}$. The number of these components is $\varkappa$. We shall call the graph $\mathcal{G}$ a connected graph, when $\varkappa = 1$.

We can find the components of a graph $\mathcal{G}$ by means of the matrix S^d. Two vertices x_i, x_j with $s_{ij}^{(d)} = 0$ belong to different components. Therefore each set of all identical rows (columns) of the stable matrix S^d of the graph $\mathcal{G}$ corresponds to one component of $\mathcal{G}$.

2. A METHOD OF TESTING THE VISIBILITY OF A POLYHEDRON

A parallel or a central projection of a finite convex polyhedron (briefly polyhedron) is a convex polygon (we exclude the case when some points of the polyhedron are projected into infinite points), [1]. The boundary of this polygon is the boundary of the polyhedron in this projection.

The set of vertices and edges of a polyhedron $\boldsymbol{M}$ can be interpreted in terms of the graph theory. Then the vertices and edges of the polyhedron $\boldsymbol{M}$ correspond to vertices and unoriented edges of a graph $\mathcal{G}$ and the relation R is generated by the list of vertices and edges of polyhedron which determines the incidence between vertices and edges.

We can interpret the projection of the set of vertices and edges of the polyhedron $\boldsymbol{M}$ in the plane by the same graph $\mathcal{G}$ by observing the following rule:

A vertex or an edge of the graph of the projection of the polyhedron will exist if and only if there exists a vertex or an edge of a graph of the polyhedron (regardless of the possible special position in the projection).

Let us assume that we can determine the boundary of the projection $\boldsymbol{M}_1$ of the polyhedron $\boldsymbol{M}$. The boundary is the set of sides and vertices of the polygon $\boldsymbol{M}_1$. We shall construct the boundary according to [1]. If two vertices of the polyhedron are projected into the same vertex of the boundary, we take into account only one of the vertices of the boundary, while the other belongs either to visible or invisible vertices. Thus the boundary generates three disjoint sets of vertices of the polyhedron:

1. V_0 – set of vertices of the boundary,
2. V_1 – set of visible vertices,
3. V_2 – set of invisible vertices.

Let us construct a subgraph $\bar{\mathcal{G}}$, the so-called factor of the graph $\mathcal{G}$, whose set of vertices coincides with that of the graph $\mathcal{G}$ while its set of edges includes only those edges of $\mathcal{G}$ that have no vertex on the boundary. The adjacency matrix $\bar{G}$ of the factor $\bar{\mathcal{G}}$ from the matrix G by replacing all elements of all rows and columns which correspond to the vertices of the boundary by 0. Let the set of vertices $V_1 \cup V_2$ and the set of edges generated by these (and no other) vertices determine a graph $\mathcal{G}^*$ as a subgraph of the factor $\bar{\mathcal{G}}$. We obtain the matrix G^* from the matrix $\bar{G}$ (or from the matrix G) by omitting those rows and columns that correspond to the vertices of the boundary.

Obviously, we can decompose the graph $\mathcal{G}^*$ into two components $\mathcal{G}^*_1$ and $\mathcal{G}^*_2$ so that $X_1 = V_1$ and $X_2 = V_2$.

The diameter $d(\mathcal{G}^*)$ of the graph $\mathcal{G}^*$ is max $[d(\mathcal{G}^*_1, d(\mathcal{G}^*_2)]$. Therefore we construct the stable matrix S^{*d} and distinguish the vertices belonging to the classes X_1 and X_2 by comparing the rows (columns).

For example, Fig. 1 shows a projection of a dodecahedron with fifteen vertices; it has the set of vertices of the boundary $V_0 = \{3, 4, 5, 10, 15, 11, 12, 13, 8\}$ and therefore the graph $\mathcal{G}^*$ has the matrix

$G^* =$

	1	2	6	7	9	14
1	0	1	1	0	0	0
2	1	0	0	1	0	0
6	1	0	0	1	0	0
7	0	1	1	0	0	0
9	0	0	0	0	0	1
14	0	0	0	0	1	0

;

this results from the matrix G of the graph $\mathcal{G}$ by omitting those rows and columns that belong to the vertices of the set V_0.

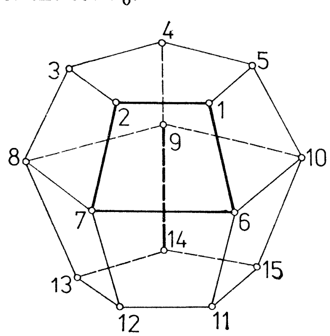

Fig. 1.

The stable matrix of the graph $\mathscr{G}^*$ is the matrix

$$S^{*2} = \begin{array}{c|cccccc} & 1 & 2 & 6 & 7 & 9 & 14 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 6 & 1 & 1 & 1 & 1 & 0 & 0 \\ 7 & 1 & 1 & 1 & 1 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 1 & 1 \\ 14 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} .$$

Vertices connected by a walk in the graph $\mathscr{G}^*$ belong to one of the two disjoint subsets $V_1 = \{1, 2, 6, 7\}$, $V_2 = \{9, 14\}$.

Let i be a visible vertex (by definition or determined in an elementary way), i.e. $i \in V_1$. The matrix G_1 arises from the matrix G in such a way that all its elements corresponding to the vertices of the set V_2 are replaced by 0. The matrix G_1 is the matrix of the graph containing all visible edges and vertices and all "neutral" edges and vertices, i.e. those which are practically drawn as "visible".

All the other edges are invisible (i.e. drawn by dashed lines) and they are determined by the matrix G_2 which fulfils $G_1 + G_2 = G$.

Note 4. When solving the visibility, the following special cases may occur:

(a) If one of the components $\mathscr{G}_1^*$ and $\mathscr{G}_2^*$ is empty and the other contains at least one vertex then $\varkappa = 1$ and all elements in the matrix S^{*d} are 1.

(b) If both these graphs are empty, then all vertices belong to the boundary.

(c) The case of the (so-called) isolated edges, i.e. edges which are not incident with the boundary but join vertices of the boundary. The visibility of an isolated edge cannot be solved by our method. Therefore we substitute the elements corresponding to the isolated edges in the matrix of the graph by 0.

Note 5. From the point of view of computer working storage, the use of Boolean matrices of a graph in the current form is not too suitable. There exist, however, various algorithms especially with regard to the operational system of the computer. Nonetheless, independently of it we can use the property that the matrix of a graph is symmetric and that the elements of its diagonal are 0. We can use this latter property e.g. in Fortrans record so that we write the variable corresponding to the element g_{ij} by the type COMPLEX and then assign the values of the first and second coordinate of the projection of the vertex x_i to its real and imaginary parts respectively.

Literature

[1] *V. Medek:* Über den Umriss der Konvexen Flächen. Aplikace matematiky 5 (1978), 378—380.

Súhrn

TESTOVANIE VIDITEL'NOSTI KONVEXNÉHO MNOHOSTENA POMOCOU GRAFOV

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Tento príspevok nadväzuje na článok [1], v ktorom sa rieši úloha o testovaní obrysu konvexného mnohostena v rovnobežnom a stredovom premietaní. Našim ciel'om je podať metódu, pomocou ktorej možno skonštruovať jednoduchý algoritmus pre automatizáciu kreslenia priemetu konvexného mnohostena v l'ubovol'nom premietaní.

V prvej časti príspevku sú uvedené niektoré pojmy z teórie grafov, ktoré sa ďalej používajú. Druhá časť je aplikáciou grafov na určenie viditeľnosti priemetu konvexného mnohostena.

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