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LATENT ROOTS OF LAMBDA-MATRICES, KRONECKER SUMS AND MATRICIAL NORMS

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1. GENERALITIES

In this paper we use the Kronecker sum $\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}$, where $\mathbf{C}$ and the unit matrix $\mathbf{I}$ are square matrices of the same order.

We shall partition the sum $\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}$ in four blocks, the diagonal blocks being square, not of the same order, and we shall take a (scalar) norm $\|\cdot\|_i$[1] $(i = 1, \infty)$ of each block. In this way, we obtain [4, 12, 13, 14, 15] a matrical norm ϕ_i, $(i = 1, \infty)$, of the referred-to sum. Then we shall calculate the spectral radius of a 2×2 non-negative matrix.

Given a lambda-matrix

$$A(\lambda) = \mathbf{I}\lambda^n + \mathbf{A}_1\lambda^{n-1} + \ldots + \mathbf{A}_{n-1}\lambda + \mathbf{A}_n \,, \tag{1.1}$$

where $\mathbf{I}, \mathbf{A}_i \in \mathbf{M}_{p,p}(\mathbf{K})$, let λ denote any latent root of $\mathbf{A}(\lambda)$, that is to say, let λ be a zero of det $\mathbf{A}(\lambda)$. We know [1, 3, 6, 7, 11] that λ is an eigenvalue of the block-companion matrix

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & -\mathbf{A}_n \\ \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} & -\mathbf{A}_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} & -\mathbf{A}_1 \end{bmatrix}.$$

We also know [2, 8] that the eigenvalues of $\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B}$ are the sums of the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$.

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[1]) For $\mathbf{B}\ (\beta_{ij}) \in \underset{r,s}{\mathrm{M}}\ (\mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we let

$$\|\mathbf{B}\|_1 : \overset{(\mathrm{def.})}{=} \underset{j=1,2\ldots,s}{\mathrm{Max}} \{\sum_{i=1}^{r} |\beta_{ij}|\}, \quad \|\mathbf{B}\|_\infty : \overset{(\mathrm{def.})}{=} \underset{i=1,2\ldots,r}{\mathrm{Max}} \{|\sum_{j=1}^{s} \beta_{ij}|\}.$$

Also needed is a result stating that, if $\phi(\boldsymbol{M})$ is a matricial norm of a matrix $\boldsymbol{M}$, then [4, 12, 13, 14, 15] $\varrho(\boldsymbol{M}) \leqq \varrho(\phi(\boldsymbol{M}))$, $\varrho(\boldsymbol{M})$ being the spectral radius of the matrix $\boldsymbol{M}$.

2. IN THIS SECTION WE DETERMINE UPPER BOUNDS FOR THE ABSOLUTE VALUES OF THE LATENT ROOTS λ OF THE LAMBDA-MATRIX (1.1)

Partitioning in the way indicated, we have

$$\text{(2.1)} \qquad \boldsymbol{C} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{C} = \left[\begin{array}{cccccc|c} \Delta\boldsymbol{C} & \Delta\boldsymbol{0} & \Delta\boldsymbol{0} & \dots & \Delta\boldsymbol{0} & & \Delta(-\boldsymbol{A}_n) \\ \Delta\boldsymbol{I} & \Delta\boldsymbol{C} & \Delta\boldsymbol{0} & \dots & \Delta\boldsymbol{0} & & \Delta(-\boldsymbol{A}_{n-1}) \\ \dots & & & & & & \dots \\ \Delta\boldsymbol{0} & \Delta\boldsymbol{0} & \Delta\boldsymbol{0} & \dots & \Delta\boldsymbol{C} & & \Delta(-\boldsymbol{A}_2) \\ \hline \Delta\boldsymbol{0} & \Delta\boldsymbol{0} & \Delta\boldsymbol{0} & \dots & \Delta\boldsymbol{I} & & \Delta(-\boldsymbol{A}_1\boldsymbol{I}) + \Delta\boldsymbol{C} \end{array}\right],$$

where $\Delta\boldsymbol{A} = \operatorname{diag}(\boldsymbol{A}, \dots, \boldsymbol{A})$ with the suitable order. (See also [17].)

Taking the (scalar) norm $\|\cdot\|_i$, $(i = 1, \infty)$, of each of the four indicated blocks, we obtain

$$\text{(2.2)} \quad \phi_i(\boldsymbol{C} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{C}) \leqq \begin{bmatrix} 1 + \|\boldsymbol{C}\|_i & \alpha_i \\ 1 & \|\boldsymbol{A}_1\|_i + \|\boldsymbol{C}\|_i \end{bmatrix} \in \mathrm{M}_{2,2}(\mathbb{R}_+), \quad (i = 1, \infty),$$

where

$$\alpha_i := \left\| \begin{matrix} \boldsymbol{A}_n \\ \boldsymbol{A}_{n-1} \\ \dots \\ \boldsymbol{A}_2 \end{matrix} \right\|_i, \quad (i = 1, \infty).$$

The matrix (2.2) is a 2-square, non-negative matrix, and we can calculate its eigenvalues. As we have

$$|\lambda| \leqq \tfrac{1}{2}\varrho[\phi_i(\boldsymbol{C} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{C})],$$

it follows that

$$\text{(2.3)} \qquad \lambda \leqq \tfrac{1}{2}\varrho\left(\begin{bmatrix} 1 + \|\boldsymbol{C}\|_i & \alpha_i \\ 1 & \|\boldsymbol{A}_1\|_i + \|\boldsymbol{C}\|_i \end{bmatrix}\right), \quad (i = 1, \infty),$$

where

$$\alpha_i = \left\| \begin{matrix} A_n \\ \dots \\ \boldsymbol{A}_2 \end{matrix} \right\|_i, \quad (1 = 1, \infty).$$

2.1. Numerical Example

Let us take the lambda-matrix

$$\boldsymbol{A}(\lambda) = \boldsymbol{I}\lambda^3 + \boldsymbol{A}_1\lambda^2 + \boldsymbol{A}_2\lambda + \boldsymbol{A}_3,$$

where

$$A_1 = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -19 & -57 \\ 0 & -76 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 20 & 140 \\ 0 & 160 \end{bmatrix}.$$

We have

$$\|A_1\|_\infty = 6, \quad \|A_2\|_\infty = 76, \quad \|A_3\|_\infty = 160, \quad \|C\|_\infty = 160$$

and

$$|\lambda| \leqq \tfrac{1}{2}\varrho \left(\begin{bmatrix} 1 + 160 & 160 \\ 1 & 4 + 160 \end{bmatrix} \right)$$

implies

$$|\lambda| \leqq 87{\cdot}62,$$

which is a much better result than that given by $|\lambda| \leqq \|C\|_\infty$ (in our case).

Remark. The bounds given by (2.3) can be better than other bounds. For example, we have the bound [16]

$$|\lambda| \leqq \operatorname*{Max}_{i=1,\dots,n} \left\{ \frac{\|A_{i+1}\|_\infty}{\|A_i\|_\infty} \right\} + \|A_1\|_\infty.$$

For the above numerical example one obtains

$$|\lambda| \leqq 89{\cdot}5.$$

2.2. Particular case: polynomials with complex coefficients.

For the polynomial

$$p(z) = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n, \quad a_i \in \mathbb{C}, \quad (i = 1, 2, \dots, n),$$

we obtain using the same argument, C being the companion matrix of $p(z)$:

$$(2.2.1) \quad \phi_i(C \otimes I + I \otimes C) \leqq \begin{bmatrix} 1 + \|C\|_i & \alpha_i \\ 1 & |a_1| + \|C\|_i \end{bmatrix} \in M_{2,2}(\mathbb{R}_+), \quad (i = 1, \infty),$$

where

$$\alpha_i := \left\| \begin{matrix} a_n \\ a_{n-1} \\ \cdots \\ a_2 \end{matrix} \right\|_i, \quad (i = 1, \infty).$$

As

$$|z| \leqq \tfrac{1}{2}\varrho[\phi_i(C \otimes I + I \otimes C)],$$

it follows that

$$(2.2.2) \qquad |z| \leqq \tfrac{1}{2}\varrho\left(\begin{bmatrix} 1 + \|C\|_i & \alpha_i \\ 1 & |a_1| + \|C\|_i \end{bmatrix} \right), \quad (i = 1, \infty)$$

with

$$\alpha_i = \left\| \begin{matrix} a_n \\ a_{n-1} \\ \cdots \\ a_2 \end{matrix} \right\|_i , \quad (i = 1, \infty) .$$

Remark. The bound given by (2.2.2) can be better than other known ones. For example, for the polynomial

$$p(z) = z^3 - 3z^2 + z - 2$$

we have

$$\phi_1(\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}) \leqq \begin{bmatrix} 7 & 3 \\ 1 & 9 \end{bmatrix}$$

and

$$\phi_\infty(\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}) \leqq \begin{bmatrix} 5 & 2 \\ 1 & 7 \end{bmatrix}$$

which by (2.2.2) yields

$$|z| \leqq 3{\cdot}86 .$$

Using the following inequalities for zeros of the polynomial $p(z) = z^n + a_1 z^{n-1} + \ldots$ $\ldots + a_{n-1} z + a_n$:

Deutsch [4]:

$$|z| \leqq \max \{2, |a_n| + |a_1|, |a_{n-1}| + |a_1|, \ldots, |a_2| + |a_1|\} ;$$

Cauchy [4, 9]:

$$|z| \leqq \max \{|a_n|, 1 + |a_1|, 1 + |a_2|, \ldots, 1 + |a_{n-1}|\} ;$$

Walsh [10, p. 221]:

$$|z| \leqq |a_1| + |a_2|^{1/2} + |a_3|^{1/3} + \ldots + |a_n|^{1/n} ;$$

Kojima [4; 9; 10, p. 221]:

$$|z| \leqq \max \left\{ \left|\frac{a_n}{a_{n-1}}\right|, 2\left|\frac{a_{n-1}}{a_{n-2}}\right|, 2\left|\frac{a_{n-2}}{a_{n-3}}\right| \ldots 2\left|\frac{a_2}{a_1}\right|, 2|a_1| \right\} ;$$

Carmichael and Mason [15; 10; p. 222]:

$$|z| \leqq (1 + \sum_{j=1}^{n} |a_j|^2)^{1/2} ;$$

– we have, respectively: $|z| \leqq 5{\cdot}00$; $|z| \leqq 4{\cdot}00$; $|z| \leqq 5{\cdot}41$; $|z| \leqq 6{\cdot}00$; $|z| \leqq 3{\cdot}87$.

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Souhrn

LATENTNÍ KOŘENY LAMBDA-MATIC, KRONECKEROVY SOUČTY A MATICOVÉ NORMY

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Pomocí Kroneckerových součtů a maticových norem je podána metoda určení horní meze pro $|\lambda|$, kde λ je latentní kořen lambda-matice. Speciálně jsou dány horní meze pro $|z|$, kde z je kořen polynomu s komplexními koeficienty. Výsledky jsou porovnány s jinými známými mezemi pro $|z|$.

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