Václav Alda

Remark on two papers concerning axiomatics of quantum mechanics

Aplikace matematiky, Vol. 25 (1980), No. 6, 453--456

Persistent URL: http://dml.cz/dmlcz/103884

REMARK ON TWO PAPERS CONCERNING AXIOMATICS OF QUANTUM MECHANICS

VÁCLAV ALDA

(Received November 10, 1978)

The use of Hilbert space for formulation of quantum mechanics [1] has led inevitably to the question what are the "physical" reasons for the usefulness or even necessity of Hilbert space in the construction of a scheme of quantum mechanics.

This problem was raised by von Neumann who has developed two ways for its solution. The first studies the algebraic structure of the set of observables, the second studies the set of yes-no experiments. Both the methods have been combined in the paper [2], where a bibliography to this problem can be found.

The task is to demonstrate:

1) There is a unit vector in Hilbert space (which is determined perhaps up to multiplication by a complex unit) for every irreducible state of a physical system, a projection for every yes – no experiment and a self-adjoint operator for every observable.

2) The probability of finding the system in the state ψ is $|(\varphi, \psi)|^2$ if the system is in the state φ (φ, ψ are the unit vectors corresponding by 1) to the states).

3) Every projector (self-adjoint operator) in Hilbert space corresponds to a yes-no experiment (to an observable). Every unit vector corresponds to a physically realizable state.

The existence of a Hilbert space related to a physical system is proved usually by exploiting the theorem about realization of a modular lattice by subspaces of a vector space. This solves 1). The problem of transition probability 2) is solved by Gleason's theorem. For the proof of this one the first part of 3) is necessary, namely, that every projector is physically meaningful (it suffices to suppose less, but something is necessary). The second part of 3) is used for the demonstration of Wigner's theorem about symmetry.

However, Gudder [3] expressed the opinion that this goal has not yet been achieved: in order to demonstrate 1) it is necessary to suppose completeness and atomicity of the lattice of yes-no experiments and these properties are not obvious. Therefore, another method for the construction of a Hilbert space for a physical system is given in [3], namely, through localization.

However, the Hilbert space constructed in [3] contains all states, not only the irreducible ones, and so it should be identified with the space of self-adjount operators A with $\operatorname{tr}(A^2) < \infty$ in the usual formulation.

The aim of this remark is to show that the construction of this Hilbert space for systems which are not localizable can be achieved by adding two axioms to the part of the system from [2] – axioms which have an evident meaning. For this modified system of axioms, atomicity is not necessary.

We accept axioms A.1 – A.9 from [2] with this modification: we shall suppose that the set of states is strongly order-determining [4].

From A.10 we take the following part:

B.1. To every irreducible state S there is an observable $\varphi(S)$. This correspondence can be extended linearly on to the linear space $\mathscr{L}$ generated by irreducible states so that the images belong to the space generated by observables and

$$0 \leqq \langle \varphi(S), \sigma \rangle \leqq 1\,, \quad S \in \mathscr{L}, \quad \sigma \text{ any state,} \tag{1}$$

is the probability of finding the system in the state S if it is in the state σ.

In [2], it is assumed that $\varphi(S)$ is a yes-no experiment for the irreducible state S. This will not be used and from (1) it can be shown that the spectrum of $\varphi(S)$ is in $\langle 0, 1 \rangle$ for every irreducible state S.

Proof. If a part of spectrum is in $(-\infty, 0)$, then $x = \varphi^{-1}((-\infty, 0)) \neq 0$ and there is a state σ with $\sigma(x) = 1$. Then $\langle \varphi(S), \sigma \rangle < 0$ which contradicts (1).

For a part of spectrum in $(1, \infty)$ the proof is similar with $x = \varphi^{-1}((1, \infty))$ and $\langle \varphi(S), \sigma \rangle > 1$.

The observable $\varphi(S)$ will be further assumed to satisfy B.2:

$$\langle \varphi(S), \sigma \rangle = \langle \varphi(\sigma), S \rangle\,, \quad S, \sigma \in \mathscr{L}\,. \tag{2}$$

This can be demonstrated ([2]), but the demonstration requires further assumptions about φ. Since (2) has a simple meaning, it is easier to assume (2) instead of other additional properties.

Before we formulate the last assumption, we introduce the following notation:

If $S_1, \ldots, S_n$ are irreducible states and λ_i non-negative numbers with $\lambda_1 + \ldots + \lambda_n = 1$, then $\sigma = \lambda_1 S_1 + \ldots + \lambda_n S_n$ is a state $\in \mathscr{L}$. $\langle \varphi(\sigma), \sigma \rangle$ is a quadratic form in $\lambda_1, \ldots$ and represents the probability which can be measured by altering the procedures for $\varphi(S_1), \ldots$ with frequencies $\lambda_1, \ldots$. It is obvious that by expanding the number of states this probability must diminish. Hence B.3. If $\sigma = \lambda_1 S_1 + \ldots + \lambda_n S_n$, $\lambda_i \geqq 0 \sum \lambda_i = 1$, if $S_1, \ldots, S_n$ are independent states and the irreducible state S_0 does not belong to the space generated by $S_1, \ldots, S_n$ then for sufficiently small $\lambda_0 > 0$ it is

$$\langle\varphi(\lambda_0 S_0 + (1 - \lambda_0)\,\sigma),\ \lambda_0 S_0 + (1 - \lambda_0)\,\sigma\rangle \leqq \langle\varphi(\sigma), \sigma\rangle\,.$$

This inequality remains true if we take the state S_0 as a linear combination of irreducible states which does not belong to the space generated by $S_1, \ldots, S_n$.

This principle of diminishing probability implies

$$\langle\varphi(\sigma), \sigma\rangle \geqq 0 \quad \text{for all} \quad \sigma \in \mathscr{L}\,. \tag{3}$$

Remark. This is true for σ in the convex cone generated by the set of irreducible states due to (1).

Proof. a) If S_1, S_2 are two distinct states, $S = \lambda S_1 + (1 - \lambda) S_2$ is a state only for λ from a finite interval. Indeed, there must be a yes-no experiment x with $S_1(x) \neq S_2(x)$, $S(x) = \lambda(S_1(x) - S_2(x)) + S_2(x)$ and this number must be $\geqq 0$ and $\leqq 1$.

b) Let S_1, S_2 be convex combinations of irreducible states $\sigma_1, \ldots, \sigma_n$. On the line connecting S_1, S_2 there are two states S_1', S_2' such that all states are on the segment with the end-points S_1', S_2'.

Let $\sigma_1, \ldots, \sigma_k$ be linearly independent. Then $\sigma_1, \ldots, \sigma_n$ are in a $(k - 1)$-dimensional hyperplane P of the k-dimensional space R_k — this follows from the fact that λS, S a state, is a state only for $\lambda = 1$.

Now S_1', S_2' are linear combinations of some states $\sigma_1, \ldots, \sigma_n$ and S_2' must lie outside the minimal subspace to which S_1' belongs and vice versa.

Let us consider the quadratic form

$$\psi(\lambda) = \langle\varphi(\lambda S_1' + (1 - \lambda) S_2'), \lambda S_1' + (1 - \lambda) S_2'\rangle \quad \text{for} \quad \lambda \in \langle 0, 1\rangle\,.$$

It is non-negative (due to (1)) and at the end-points it is decreasing by our assumption B.3. It is therefore non-negative for all λ.

c) Let $S = \lambda_1\sigma_1 + \ldots + \lambda_j\sigma_j - (\lambda_{j+1}\sigma_{j+1} + \ldots + \lambda_n\sigma_n)$ be any combination of irreducible states $\sigma_1, \ldots, \sigma_n$ and all $\lambda_i \geqq 0$.

If we suppose $\lambda_1 + \ldots + \lambda_j + \lambda_{j+1} + \ldots + \lambda_n = 1$ then $S = \lambda(\lambda_1'\sigma_1 + \ldots + \lambda_j'\sigma_j) + (1 - \lambda)(\lambda_{j+1}\sigma_{j+1} + \ldots \lambda_n'\sigma_n)$ and $\lambda_1'\sigma_1 + \ldots, \lambda_{j+1}\sigma_{j+1} + \ldots$ are two states and we can apply b).

If $\varkappa = \lambda_1 + \ldots + \lambda_n \neq 0$, we can multiply by $\varkappa^{-1}$.

If $\lambda_1 + \ldots + \lambda_n = 0$, we form $\varepsilon\sigma_1 + \lambda_{i1}\sigma_1 + \ldots + \lambda_n\sigma_n$. It is $\langle\varphi(\varepsilon\sigma_1 + \lambda_1\sigma_1 + \ldots), \varepsilon\sigma_1 + \ldots\rangle = \varepsilon^2\langle\ldots\rangle + \varepsilon\{\ldots\} + \langle\varphi(\lambda_1\sigma_1 + \ldots), \lambda_1\sigma_1 + \ldots\rangle$.

This is $\geqq 0$ for $\varepsilon > 0$ and the last term on the right hand side must be $\geqq 0$.

Now (2) and (3) state that the space $\mathscr{L}$ equipped with the bilinear form

$$(\sigma, S) = \langle\varphi(\sigma), S\rangle$$

is a pre-Hilbert space over the field of real numbers.

References

[1] *J. von Neumann:* Mathematische Grundlagen der Quantenmechanik. Springer - Verlag, 1932.
[2] *J. Gunson:* On the Algebraic Structure Of Quantum Mechanics. Comm. Math. Phys., 6 (1967), 262—285.
[3] *S. Gudder:* On the Quantum Logic Approach to Quantum Mechanics. Comm. Math. Phys., 12 (1969), 1—15.
[4] *J. C. T. Pool:* Baer *-Semigroups and the Logic of Quantum Mechanics. Comm. Math. Phys. 9 (1968), 118—141.

Souhrn

POZNÁMKA KE DVĚMA ČLÁNKŮM O AXIOMATICE KVANTOVÉ MECHANIKY

VÁCLAV ALDA

Modifikací schématu (pocházejícího od Gunsona) lze ukázat, že prostor generovaný všemi irreducibilními stavy má prehilbertovskou strukturu. Positivita skalárního součinu plyne z předpokladu, že pravděpodobnost nalezení systému v irreducibilním stavu klesá s rostoucím počtem složek stavu, v němž se systém nachází.

Author's address: Doc. Dr. *Václav Alda*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.