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An application of the induction method of V. Pták to the study of regula falsi


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Let $\mathbb{R}^p$ denote the $p$-dimensional Euclidean space, i.e. the set of all vectors $r$ of $p$ real components $r_i$, $i = 1, \ldots, p$. In $\mathbb{R}^p$ we shall consider the relation "≤" induced by the cone $\mathbb{R}^p_+ = \{ r \in \mathbb{R}^p; r_i \geq 0, i = 1, \ldots, p \}$. Let $T$ be a $p$-dimensional interval of the form $T = \{ r \in \mathbb{R}^p; 0 < r_i < a_i \} i = 1, 2, \ldots, p$, where the numbers $a_i i = 1, \ldots, P$, are supposed to satisfy the inequalities $a_1 \geq a_2 \geq \ldots \geq a_p < 0$. Some of these numbers may be infinite. If $\omega$ is a real function of $p$ real variables which maps the $p$-dimensional interval $T$ into the interval $]0, a_p[ = \{ r \in \mathbb{R}; 0 < r < a_p \}$, then we may define inductively:

\begin{equation}
\omega^0(r_1, \ldots, r_p) = r_p, \omega^{n+1}(r_1, \ldots, r_p) = \omega^n(r_2, \ldots, r_p, \omega(r_1, \ldots, r_p))
\end{equation}

$n = 0, 1, 2, \ldots; \; r \in T$.

Using the above notation we can define the notion of $p$-dimensional rate of convergence which generalizes the notion of rate of convergence given in [5], [6].

**Definition 1.** A function $\omega : T \to ]0, a_p[$ is called a $p$-dimensional rate of convergence if the series $\sigma(r) = \sum_{n=0}^{\infty} \omega^n(r)$ is convergent for any $r \in T$.

To the $p$-dimensional rate of convergence $\omega$, let us attach a vector function $\omega : T \to T$ defined by

\begin{equation}
\omega(r_1, \ldots, r_p) = (r_2, \ldots, r_p, \omega(r_1, \ldots, r_p)), \; r \in T.
\end{equation}

Then the iterates $\omega^n$ of $\omega$ are defined as follows:

\begin{equation}
\omega^0(r) = r, \quad \omega^{n+1}(r) = \omega(\omega^n(r)), \quad n = 0, 1, 2, \ldots, r \in T.
\end{equation}

As $\omega$ is a $p$-dimensional rate of convergence, then for any $r \in T$, the following series is convergent in $\mathbb{R}^p$:

\begin{equation}
\sigma(r) = \sum_{n=0}^{\infty} \omega^n(r).
\end{equation}
The above introduced vector functions \( \omega \) and \( \sigma \) are evidently connected by the following relation:

\[
\sigma(r) = \sigma(\omega(r)) + r, \quad r \in T.
\]

Let us denote by \( \sigma_1, \ldots, \sigma_p \) the components of the vector function \( \sigma \). Then we obviously have the following relations:

\[
\sigma_p(r) = \sigma(r) = \sum_{n=0}^{\infty} \omega^n(r), \quad r \in T,
\]
\[
\sigma_{p-k}(r) = \sigma(r) + \sum_{j=p-k}^{\infty} r_j, \quad k = 1, 2, \ldots, p-1, \quad r \in T.
\]

Now, let \((X, d)\) be a complete metric space, \( x \) an element of \( X \), and \( A \) a subset of \( X \). We denote by \( d(x, A) \) the g.l.b. of the set \( \{d(x, y); y \in A\} \). If \( A \) is a subset of \( X^p \), then \( A_i \) will denote the set \( \{x_i \in R; (x_1, \ldots, x_p) \in A\} \). Let \( x = (x_1, \ldots, x_n) \) be an element of \( X^p \). We denote by \( d(x, A) \) the vector from \( R^p_+ \) with components \( d(x_i, A_i), i = 1, 2, \ldots, p \). We shall use the following notation:

\[
U(A; r) = \{x \in X^p; d(x, A) \leq r\}.
\]

If \( x \) is an element of \( X^p \), we shall write \( U(x, r) \) instead of \( U(\{x\}, r) \) for simplicity. Let \( \{Z(r)\}_{r \in T} \) be a family of subsets of the space \( X^p \). We define the limit of this family by

\[
Z(0) = \bigcap_{r \in T} (\bigcup_{s \leq r} Z(s))^{-}.
\]

We can state now the following generalisation of the Induction Theorem of V. Pták [4]:

**Theorem 1.** If

\[
Z(r) \subset U(Z(\omega(r)), r),
\]

for each \( r \in T \), then

\[
Z(r) \subset U(Z(0), \sigma(r))
\]

for each \( r \in T \).

**Proof.** If \( x_0 \in Z(r) \), then by (10) there exists an \( x_1 \in U(x_0, r) \cap Z(\omega(r)) \). Now, using again (10), there exists an \( x_2 \in U(x_1, \omega(r)) \cap Z(\omega^2(r)) \). We infer by induction that for any \( n \in \{0, 1, 2, \ldots\} \), there exists an \( x_{n+1} \in U(x_n, \omega^n(r)) \cap Z(\omega^{n+1}(r)) \). Because of the convergence of the series (4) it follows that the sequence \( (x_n)_{n=1}^{\infty} \) is a Cauchy sequence in \( X^p \). Hence it has a limit \( x_\infty \). Now, as \( x_n \in Z(\omega^n(r)) \) and \( \lim_n \omega^n(r) = 0 \) it follows that \( x_\infty \in Z(0) \).

On the other hand, \( d(x_0, x_\infty) \leq \sum_{n=0}^{\infty} d(x_{n+1}, x_n) \leq \sum_{n=0}^{\infty} \omega^n(r) = \sigma(r) \) and thus the proof of the theorem is complete. \( \square \)
In the following we shall show how Theorem 1 can be applied to the study of convergence of iterative procedure of the form

\[ x_{n+1} = F(x_{n-p+1}, x_{n-p+2}, \ldots, x_n), \quad n = 0, 1, 2, \ldots, \]

where \( F \) is a mapping of \( X^p \) into \( X \) and \( x_0 = (x_{-p+1}, x_{-p+2}, \ldots, x_0) \) is a fixed element of \( X^p \). Suppose we can attach to the pair \( (F, x_0) \) a family of sets \( \{Z(r)\}_{r \in T} \subset X^p \) and a \( p \)-dimensional rate of convergence \( \omega \) such that the following conditions are satisfied:

\[ x_0 \in Z(r_0) \quad \text{for a certain } r_0 \in T. \]

\[ \text{If } r \in T \text{ and } y = (y_1, \ldots, y_p) \in Z(r), \text{ then} \]

\[ (y_2, \ldots, y_p, F(y)) \in Z(\omega(r)) \cap U(y, r). \]

The above conditions imply, according to Theorem 1, that \( Z(0) \) is not void. Moreover, it follows that via the iterative procedure (12) one obtains a sequence \( \{x_n\}_{n=0}^{\infty} \) which converges to an element \( x^* \in X \) with \( (x^*, \ldots, x^*) \in Z(0) \), and such that for any \( n \in \{0, 1, 2, \ldots\} \) the following relations are satisfied:

\[ x_n = (x_{n-p+1}, x_{n-p+2}, \ldots, x_n) \in Z(\omega^n(r_0)). \]

\[ d(x_{n+1}, x_n) \leq \omega^n(r_0). \]

\[ d(x_n, x_0) \leq \sigma(r_0) - \sigma(\omega^n(r_0)). \]

\[ d(x_n, x^*) \leq \sigma(\omega^n(r_0)). \]

The inequality (18) will be called an apriori estimate of the distance between the elements of the sequence \( \{x_n\}_{n=0}^{\infty} \) and \( x^* \). The name of “apriori estimate” is justified by the fact that the right hand side of (18) can be computed before obtaining \( x_1, x_2, \ldots, x_n \) via the iterative procedure. Let us now suppose that for a certain \( n \in \{1, 2, \ldots\} \) one has already obtained \( x_1, x_2, \ldots, x_n \). If

\[ x_{n-1} \in Z(d(x_n, x_{n-1})) \]

then taking \( x_{n-1} \) instead of \( x_0 \) and \( d(x_n, x_{n-1}) \) instead of \( r_0 \), we infer, like in (18), that

\[ d(x_n, x^*) \leq \sigma(\omega(d(x_n, x_{n-1})) = \sigma(d(x_n, x_{n-1})) - d(x_n, x_{n-1}). \]

This inequality is called an “aposteriori estimate”, because it can be computed only after obtaining \( x_1, x_2, \ldots, x_n \) via the iterative procedure.

Summing up what we have stated above, we obtain the following.

**Corollary 1.** If the conditions (13) and (14) hold, then via the iterative procedure (12) one obtains a sequence \( \{x_n\}_{n=0}^{\infty} \) which converges to an element \( x^* \in X \) such that the relations (15)–(18) are satisfied. If, in addition, for some \( n \in \{1, 2, \ldots\} \) the condition (19) is fulfilled, then for this \( n \) the inequality (20) is also satisfied.
In the sequel we shall apply Corollary 1 to the study of convergence of the secant method. First we shall state a lemma, whose proof is mainly based on the convergence of the secant method in a very particular case:

**Lemma 1.** If \( d, H, q_0 \) and \( r_0 \) are positive constants satisfying the inequality

\[
(\sqrt{(r_0)} + \sqrt{(q_0 + r_0)})^2 \leq \frac{d}{H},
\]

then the function

\[
\omega(q, r) = \frac{r(q + r)}{r + 2\sqrt{(r(q + r) + a^2)}}
\]

is a 2-dimensional rate of convergence on the interval \( T = \{(q, r); 0 < q < \infty, 0 < r < \infty\} \), and the corresponding function \( \sigma \) is given by

\[
\sigma(q, r) = r - a + \sqrt{(r(q + r) + a^2)},
\]

where

\[
a = \frac{1}{2H} \sqrt{((d - Hq_0)^2 - 4Hdr_0)}.
\]

**Proof.** First let us remark that the condition (22) implies that \((d - Hq_0)^2 \geq 4Hdr_0\), so that the formula (25) makes sense. Let us consider the real polynomial

\[
f(x) = H(x^2 - a^2)
\]

and let us denote by \( x^* = a \) its positive root. Let us consider for each pair \((q, r) \in T\), the points

\[
x_0 = r + \sqrt{(r(q + r) + a^2)} \quad \text{and} \quad x_{-1} = x_0 + q.
\]

It is easy to prove that by the algorithm

\[
x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_{n-1}) - f(x_n)} f(x_n), \quad n = 0, 1, 2, \ldots,
\]

one obtains a sequence \( \{x_n\}_{n=0}^{\infty} \) which converges to \( x^* \). We have evidently \( x_{-1} - x_0 = q \) and \( x_0 - x_1 = r \). Taking \( \omega(q, r) = \left[ (x_0 - x_1)/(f(x_0) - f(x_1)) \right] f(x_1) \) and \( \sigma(q, r) = x_0 - x^* \), one obtains exactly the expressions (23) and (24). The fact that the series \( \sum_{n=0}^{\infty} \omega_n(q, r) \) is convergent and that its sum equals \( \sigma(q, r) \) is obvious. Moreover, for any \( n \in \{0, 1, 2, \ldots\} \) we have:

\[
x_n - x_{n+1} = \omega_n(q, r),
\]

\[
x_n - x_0 = \sigma(q, r) - \sigma(\omega_n(q, r)),
\]

\[
x_n - x^* = \sigma(\omega_n(q, r)).
\]
where, as in (5), we have denoted

(31) \( \omega^0(q, r) = (q, r), \quad \omega^n(q, r) = (\omega^{n-1}(q, r), \quad \omega^n(q, r), \quad n = 1, 2, \ldots \)

The generalization of the secant method that we will study below is based on the notion of divided difference of an operator. This notion was introduced by J. Schröder [8] and represents a generalization of the usual notion of divided difference of a function [3], in the same sense in which the Fréchet derivative [2] represents a generalization of the classical notion of derivative.

Let \( f \) be a (nonlinear) operator which maps a Banach space \( E \) into a Banach space \( F \) and let \( x \) and \( y \) be two distinct points of its domain. Let us denote by \( L(E, F) \) the Banach space of all bounded linear operators defined on \( E \) and with values in \( F \).

**Definition 2.** A linear operator \( A \in L(E, F) \) is called a divided difference of the operator \( f \) on the points \( x \) and \( y \), if the following equality holds:

(32) \[ A(x - y) = f(x) - f(y). \]

Concerning the existence of the divided differences of an operator, see [1]. Concerning examples in some particular spaces, see [10].

Using the above defined notion A. Sergeev [9], and J. Schmidt [7] generalized the secant method, obtaining an iterative procedure for solving nonlinear equations in Banach spaces. Let the closed sphere \( U = U(x_0, m) \) be included in the domain of \( f \) and let \( D \) denote the set \( \{(x, y) \in U \times U; x \neq y\} \). We may consider a mapping \( D \ni (x, y) \to [x, y; f] \in L(E, F) \), where \( [x, y; f] \) represents a divided difference of the operator \( f \) at the points \( x \) and \( y \), i.e.

(33) \[ [x, y; f](x - y) = f(x) - f(y). \]

In [9] the author supposes that the mapping \( (x, y) \to [x, y; f] \) is symmetric (i.e., \( [x, y; f] = [y, x; f] \)), while in [7] this assumption is no longer required. In both of the above cited papers one supposes that the mapping \( (x, y) \to [x, y; f] \) satisfies a Lipschitz condition. We shall write this condition in the form

(34) \[ \|[x, y; f] - [u, v; f]\| \leq H(x - u) + |y - v|. \]

It is easy to prove that if the above inequality holds for all \( x, y, u, v \) of \( U \) with \( x \neq y \) and \( u \neq v \), then the limit \( \lim_{y \to x} [x, y; f] \) exists for any \( x \in U \), and it equals the Fréchet derivative \( f'(x) \). Thus the mapping \( (x, y) \to [x, y; f] \) can be extended from \( D \) to \( U \times U \) by taking \( [x, x; f] = f'(x) \). Let now \( x_{-1} \) be a point of \( U \) such that the divided difference \( [x_{-1}, x_0; f] \) is boundedly invertible. The generalized secant method is described by the following algorithm:

(35) \[ x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n), \quad n = 0, 1, 2, \ldots \]

The above iterative procedure makes sense if at each step the operator \( [x_{n-1}, x_n; f] \)
is invertible and the point \(x_{n+1}\) obtained lies in the domain of \(f\). In the following, we shall apply Corollary 1 and Lemma 1 to the study of convergence of the iterative procedure (35). We shall give sufficient conditions for the convergence of the sequence \(\{x_n\}_{n=0}^\infty\) to a root \(x^*\) of the equation \(f(x) = 0\), and shall obtain sharp estimates for the distances \(\|x_n - x^*\|\).

**Theorem 2.** If the conditions (33) and (34) are satisfied for all \(x, y, u, v \in U = U(x_0, m)\) and if the following inequalities are fulfilled:

\[
\begin{align*}
(36) & \quad \|x_0 - x_{-1}\| \leq q_0, \\
(37) & \quad (\|x_{-1}, x_0; f\|^{-1})^{-1} \geq d, \\
(38) & \quad \|x_{-1}, x_0; f\|^{-1} f(x_0) \leq r_0, \\
(39) & \quad m \geq \sigma(q_0, r_0), \\
(40) & \quad (\sqrt{(r_0)} + \sqrt{(q_0 + r_0)})^2 \leq \frac{d}{H},
\end{align*}
\]

then the iterative procedure (35) makes sense and the sequence \(\{x_n\}_{n=0}^\infty\) obtained by it converges to a root \(x^*\) of the equation \(f(x) = 0\), so that the following inequalities hold:

\[
\begin{align*}
(41) & \quad \|x_n - x_0\| \leq \sigma(q_0, r_0) - \sigma(\omega(q_0, r_0)), \quad n = 0, 1, 2, \ldots, \\
(42) & \quad \|x_n - x^*\| \leq \sigma(\omega(q_0, r_0)), \quad n = 0, 1, 2, \ldots, \\
(43) & \quad \|x_n - x^*\| \leq \sigma(\|x_{n-1} - x_{n-2}\|, \|x_n - x_{n-1}\|) - \|x_n - x_{n-1}\|, \\
& \quad n = 1, 2, \ldots,
\end{align*}
\]

where \(\omega\) and \(\sigma\) are given by (23) and (24) and \(\omega\) is related to \(\omega\) as in (31).

**Proof.** For any pair of positive numbers \((q, r)\) we consider the set

\[
Z(q, r) = \{(x, y) \in U \times U; \|x - y\| \leq q, \|y - x_0\| \leq \sigma(q_0, r_0) - \sigma(q, r), \\
\|x, y; f\|^{-1} \geq h(q, r), \|x, y; f\|^{-1} f(y) \leq r \};
\]

where we denote

\[
h(q, r) = 2a + H(q + 2\sigma(q, r)).
\]

It is easy to verify that \(h(q_0, r_0) = d\). This relation together with the inequalities (36)–(39) implies that \((x_{-1}, x_0) \in Z(q_0, r_0)\). Thus the condition (13) of Corollary 1 is fulfilled. Now, let us suppose that \((x, y) \in Z(q, r)\). Denoting

\[
z = y - [x, y; f]^{-1} f(y),
\]

we have to prove that \((y, z) \in Z(r, \omega(q, r))\). The condition \(\|z - y\| \leq r\) is obvious.
Taking into account the fact that (see (5))

\[ \sigma(q, r) - r = \sigma(r, \omega(q, r)) \]  

we infer that \( \|z - x_0\| \leq \sigma(q_0, r_0) - \sigma(r, \omega(q, r)) \).

This relation implies that \( z \) belongs to \( U \). In order to prove the invertibility of 
\([y, z; f]\), we shall use the fact if \( A \) and \( B \) are two linear operators belonging to \( \mathcal{L}(E, F) \) such that \( A \) is boundedly invertible and \( \|A - B\| < \|A^{-1}\|^{-1} \), then \( B \) is also boundedly invertible and \( \|B^{-1}\|^{-1} \geq \|A^{-1}\|^{-1} - \|A - B\| \). According to (34), (44) and (45) we have

\[ \| [x, y; f] - [y, z; f] \| \leq H(q + r) < h(q, r) \leq \| [x, y; f] \|^{-1} \]  

so that \([y, z; f]\) is boundedly invertible and we have

\[ \| [y, z; f] \|^{-1} \geq h(q, r) - H(q + r) = h(r, \omega(q, r)) \]  

From (46) we infer that

\[ f(z) = f(z) - f(y) - [x, y; f] (z - y) = ([z, y; f] - [x, y; f]) (z - y) \]  

Using (34), (45) and (48), from the above equality we obtain

\[ \| [y, z; f] \|^{-1} f(z) \| \leq \| h(r, \omega(q, r)) \|^{-1} H(q + r) r = \omega(q, r) , \]

and the proof of the fact that \((y, z) \in Z(r, \omega(q, r))\) is complete. Thus the condition
(14) of Corollary 1 is also fulfilled. According to this Corollary the sequence \( \{x_n\}_{n=0}^\infty \) obtained by (35) converges to a point \( x^* \) so that the inequalities (41) and (42) hold (see (17) and (18)). As in (49) we infer that

\[ f(x_{n+1}) = ([x_{n+1}, x_n; f] - [x_{n-1}, x_n; f]) (x_{n+1} - x_n) \]

and, passing to the limit, we obtain \( f(x^*) = 0 \).

In order to complete the proof of the theorem, we still have to demonstrate the inequality (43). For this purpose, according to Corollary 1, it is sufficient to prove that

\[ (x_{n-2}, x_{n-1}) \in Z(\|x_{n-2} - x_{n-1}\|, \|x_{n-1} - x_n\|) , \]

for every \( n \in \{1, 2, \ldots\} \) (see (19)). The first and the last condition from the definition
(44) of \( Z(q, r) \) are obviously fulfilled in this case. From (24) and (45) it follows that the functions \( \sigma \) and \( h \) are increasing in the sense that if \( q \leq q_1 \) and \( r \leq r_1 \), then \( \sigma(q, r) \leq \sigma(q_1, r_1) \) and \( h(q, r) \leq h(q_1, r_1) \). According to (15) we have

\[ \|x_{n-2} - x_{n-1}\| \leq \omega^{n-2}(q_0, r_0) \quad \text{and} \quad \|x_{n-1} - x_n\| \leq \omega^{n-1}(q_0, r_0) \]

for \( n = 1, 2, \ldots, \)

where for \( n = 1 \) one has to take \( \omega^{-1}(q_0, r_0) = q_0 \).
The above inequalities imply that
\begin{align*}
\sigma(\|x_{n-2} - x_{n-1}\|, \|x_{n-1} - x_n\|) &\leq \sigma(\omega^{n-1}(q_0, r_0)) \quad \text{and} \\
h(\|x_{n-2} - x_{n-1}\|, \|x_{n-1} - x_n\|) &\geq h(\omega^{n-1}(q_0, r_0)), \quad n = 1, 2, \ldots
\end{align*}

From (15) it follows that \((x_{n-2}, x_{n-1}) \in Z(\omega^{n-1}(q_0, r_0))\) for \(n = 1, 2, \ldots\), so that we have
\begin{align*}
\|x_{n-1} - x_0\| \quad \leq \quad \sigma(q_0, r_0) - \sigma(\omega^{n-1}(q_0, r_0)) &\quad \text{and} \\
\|x_{n-2}, x_{n-1}; f\|^{-1} - 1 &\geq h(\omega^{n-1}(q_0, r_0)), \quad n = 1, 2, \ldots
\end{align*}

Finally, (51) and (52) imply that the second and the third condition of (44) are also satisfied in our case.

Let us add some remarks concerning the hypotheses of the above theorem. The constant \(q_0\) appearing in (36) can be taken as small as desired, because having an initial approximation \(x_0\), we can take \(x_{-1}\) to be a small perturbation of \(x_0\), for example \(x_{-1} = (1 + \varepsilon)x_0\). The crucial hypothesis of Theorem 2 is the inequality (40). This inequality is satisfied only if \(r_0\) is small enough, which means that the initial approximation \(x_0\) is close enough to the root \(x^*\). However, we shall show that the condition (40) is, in a sense, the weakest possible.

More precisely, we have

**Proposition 1.** For any positive constants \(d, H, q_0\) and \(r_0\) with \(H(\sqrt{(r_0)} + \sqrt{(q_0 + r_0)})^2 > d\), there exist a function \(f : \mathbb{R} \to \mathbb{R}\) and two points \(x_0\) and \(x_{-1}\) such that (34) holds for all \(x, y, u, v \in \mathbb{R}\), the conditions (36)–(38) are satisfied, but the equation \(f(x) = 0\) has no solution.

**Proof.** If \(H(\sqrt{(q_0 + r_0)} + \sqrt{(r_0)})^2 > d > H(\sqrt{(q_0 + r_0)} - \sqrt{(r_0)})^2\), take
\[f(x) = Hx^2 + dr_0 - \frac{1}{4H} (d - Hq_0)^2, \quad x_0 = \frac{d - Hq_0}{2H}, \quad x_{-1} = \frac{d + Hq_0}{2H}.
\]
If \(H(\sqrt{(q_0 + r_0)} - \sqrt{(r_0)})^2 \geq d\), take \(f(x) = (d/q_0)x^2 + r_0\), \(x_0 = 0\), \(x_{-1} = q_0\).

In the following proposition, we shall prove that the estimates (42) and (43) are in a sense, the best possible:

**Proposition 2.** For any positive constants \(d, H, q_0\) and \(r_0\) with \(H(\sqrt{(r_0)} + \sqrt{(q_0 + r_0)})^2 \leq d\) there exist a function \(f : \mathbb{R} \to \mathbb{R}\) and two points \(x_0\) and \(x_{-1}\) which satisfy the hypotheses of Theorem 2, and for which the inequalities (41)–(43) are verified with the signs of equality.

**Proof.** The proof of this proposition is a consequence of the proof of Lemma 1; indeed, for \(f\) given by (26) and \(x_0, x_{-1}\) given by (27) with \(q = q_0\) and \(r = r_0\) we have
\[\frac{f(x_{-1}) - f(x_0)}{x_{-1} - x_0} = d \quad \text{and} \quad \frac{f(x_0)}{d} = r_0.
\]
Finally, we shall try to answer the question concerning the uniqueness of the solution of the equation \( f(x) = 0 \). From (41) it follows that \( \|x^* - x_0\| \leq \sigma(r_0) \). Let \( \mathcal{V} \) denote the open sphere with center \( x_0 \) and radius \( \mu = \sigma(q_0, r_0) + 2a \).

**Proposition 3.** If the inequality (40) from Theorem 2 is strict, then the root \( x^* \), whose existence is guaranteed by the same theorem, is the unique solution of the equation \( f(x) = 0 \) in the set \( U \cap \mathcal{V} \).

**Proof.** First, we note that the inequality (40) is equivalent to the inequality
\[
\frac{d}{H} \geq (q_0 + 2r_0) + \sqrt{(r_0(q_0 + r_0))}.
\]
If either (40) or (53) is strict, then \( a > 0 \). Let \( y^* \) be an element of \( U \cap \mathcal{V} \) such that \( f(y^*) = 0 \). Using (33) we obtain the relation
\[
x^* - y^* = [x_{-1}, x_0; f]^{-1} ([x_{-1}, x_0; f] - [x^*, y^*; f]) (x^* - y^*).
\]
Now, taking into account (34) we obtain
\[
\|x^* - y^*\| \leq \frac{H}{d} (\|x^* - x_{-1}\| + \|y^* - x_0\|) \|x^* - y^*\|.
\]
On the other hand, from (24), (36) and (53) we infer that
\[
\frac{H}{d} (\|x^* - x_{-1}\| + \|y^* - x_0\|) < \frac{H}{d} (2\sigma(r_0) + 2a + q_0) = 1.
\]
Finally, the inequalities (55) and (56) imply that \( x^* = y^* \), so that the proof of the proposition is complete. □

**References**


$$x_{n+1} = (F(x_{n-p+1}, x_{n-p+2}, \ldots, x_n), \quad n = 0, 1, 2, \ldots$$

Výsledky jsou ilustrovány na příkladě konvergence metody sečení a jsou odvozeny ostré odhady pro chybu každého kroku iteračního procesu.

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