CONTACT BETWEEN ELASTIC BODIES – II. FINITE ELEMENT ANALYSIS

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INTRODUCTION

In Part I of our paper [1] the existence and uniqueness of continuous contact problems have been discussed. In the present Part we apply the simplest finite element technique, i.e., the piecewise linear triangular elements, to the solution of the contact problems. Some error estimates are deduced, assuming that the exact solution is regular enough and using a modified method of Falk. If the solution is not regular, we prove the convergence itself. For problems with enlarging contact zone, the element of Zlámal's type [3] with one curved side along the boundaries, are employed.

1. APPROXIMATION OF VARIATIONAL INEQUALITIES

Let $H$ be a real Hilbert space, $\mathcal{K} \neq \emptyset$ a closed convex subset of $H$ and $J : H \to R_1$ be a given quadratic functional:

$$J(v) = \frac{1}{2} A(v, v) - f(v),$$

where $A(u, v)$ is a symmetric, positive-semidefinite bilinear form defined on $H \times H$, $f \in H'$ a given linear continuous functional on $H$.

We shall consider the following problem:

$$(\mathcal{P}) \quad \begin{cases}
\text{find } u \in \mathcal{K} \text{ such that } \\
J(u) = \min_{v \in \mathcal{K}} J(v).
\end{cases}$$

Let $\{\mathcal{K}_h\}$, $h \in (0, 1)$ be a system of finite-dimensional approximations of $\mathcal{K}$, i.e. $\mathcal{K}_h \neq \emptyset$ are closed convex subset of $H$ for $h \in (0, 1)$, contained in finite-dimensional subspaces $S_h \subset H$. Let us define an element $u_h \in \mathcal{K}_h$ such that

$$(\mathcal{P}_h) \quad J(u_h) = \min_{v \in \mathcal{K}_h} J(v).$$
Lemma 1.1. Let $u, u_h$ be solutions of $(\mathcal{P}), (\mathcal{P}_h)$, respectively. Then it holds:

\begin{equation}
A(u - u_h, u - u_h) \leq \left\{ f(u - v_h) + f(u_h - v) + A(u_h - u, v_h - u) + \right. \nonumber \\
+ \left. A(u, v - u_h) + A(u, v_h - u) \right\}
\end{equation}

for any $v \in \mathcal{X}, v_h \in \mathcal{X}_h$.

Proof. As $J$ is convex on $H$ then $u \in \mathcal{X}, u_h \in \mathcal{X}_h$ solve $(\mathcal{P}), (\mathcal{P}_h)$ respectively if and only if

$A(u, v - u) \geq f(v - u) \quad \forall v \in \mathcal{X},$

$A(u_h, v_h - u_h) \geq f(v_h - u_h) \quad \forall v_h \in \mathcal{X}_h.$

Hence

$A(u - u_h, u - u_h) = A(u, u) + A(u_h, u_h) - 2A(u, u_h) \leq$

$\leq A(u, v) + f(u - v) - A(u_h, v_h) + f(u_h - v_h) - 2A(u, u_h) =$

$= f(u - v_h) + f(u_h - v) + A(u_h - u, v_h - u) +$

$+ A(u, v_h - u) \quad \forall v \in \mathcal{X}, \quad \forall v_h \in \mathcal{X}_h.$

Remark 1.1. If $\mathcal{X}_h \subseteq \mathcal{X}$, the inequality (1.1) yields

\begin{equation}
A(u - u_h, u - u_h) \leq \left\{ f(u - v_h) + A(u_h - u, v_h - u) + A(u, v_h - u) \right\}
\end{equation}

$\forall v_h \in \mathcal{X}_h.$

Proof. Inserting $v = u_h$ in (1.1), we obtain (1.2).

Remark 1.2. Let $\| \|$ and $\| \|$ be a norm and a seminorm in $H$, respectively. If there exists a constant $\gamma < 0$ such that

\begin{equation}
A(v, v) \geq \gamma \| v \|^2 \quad \forall v \in H,
\end{equation}

then $(\mathcal{P}), (\mathcal{P}_h)$ have unique solutions $u$ and $u_h$, respectively and

\begin{equation}
\gamma \| u - u_h \|^2 \leq \left\{ f(u - v_h) + f(u_h - v) + A(u_h - u, v_h - u) + \right. \nonumber \\
+ \left. A(u, v - u_h) + A(u, v_h - u) \right\} \quad \forall v \in \mathcal{X}, \quad \forall v_h \in \mathcal{X}_h.
\end{equation}

If

\begin{equation}
A(v, v) \geq \gamma |v|^2 \quad \forall v \in H,
\end{equation}

then $|u - u_h|$ instead of $\|u - u_h\|$ can be written in the left hand side of (1.4).

Remark 1.3. As $A(u, v)$ is symmetric and positive semidefinite, we have

$A(u, v) \leq 1/2 A(u, u) + 1/2 A(v, v) \quad \forall u, v \in H$

and (1.1) can be written as follows:

$1/2 A(u - u_h, u - u_h) \leq \left\{ f(u - v_h) + f(u_h - v) + 1/2 A(v_h - u, v_h - u) + \right. \nonumber \\
+ \left. A(u, v - u_h) + A(u, v_h - u) \right\} \quad \forall v \in \mathcal{X}, \quad \forall v_h \in \mathcal{X}_h.$
Theorem 1.1. Let us assume that

\[ (1.6) \quad \forall v \in \mathcal{K} \quad \exists \{v_h\} \in \{\mathcal{K}_h\} : \|v - v_h\| \to 0 , \quad h \to 0^+ ; \]
\[ (1.7) \quad v_h \in \mathcal{K}_h , \quad v_h \rightharpoonup v \quad \text{weakly in } H \implies v \in \mathcal{K} . \]

Let the form \( A \) satisfy (1.3). Then

\[ \|u - u_h\| \to 0 , \quad h \to 0^+ . \]

For the proof, see e.g. [2] — chpt. 4.

Remark 1.4. If \( \mathcal{K}_h \subset \mathcal{K} , \forall h \in (0, 1) \), then (1.7) is satisfied, since \( \mathcal{K} \) is weakly closed.

We shall need also a slight modification of Theorem 1.1.

Theorem 1.2. Let us suppose that

\[ (1.8) \quad \|v_h\| \to \infty , \quad v_h \in \mathcal{K}_h \implies J(v_h) \to + \infty \]

and let (1.5), (1.6), (1.7) be satisfied. Moreover, let us suppose that \( \mathcal{P} \) has precisely one solution. Then

\[ |u - u_h| \to 0 , \quad h \to 0^+ . \]

Proof. (1.6) ensures the existence of \( \{v_h\} , v_h \in \mathcal{K}_h \) such that

\[ \|v_h - u\| \to 0 , \quad h \to 0^+ . \]

Hence

\[ (1.9) \quad J(v_h) \to J(u) , \quad h \to 0^+ \]

and from the definition of \( (\mathcal{P}_h) : J(u_h) \leq J(v_h) \).

From (1.8), (1.9) the boundedness of \( \{u_h\} \) follows:

\[ \exists c = \text{const.} > 0 : \|u_h\| \leq c \quad \forall h \in (0, 1) . \]

Then there exists an element \( u^* \in H \) and a subsequence of \( \{u_h\} \) (let us denote it by \( \{u_h\} \) again) such that:

\[ u_h \rightharpoonup u^* \quad \text{in } H . \]

From (1.7) it follows that \( u^* \in \mathcal{K} \). As \( J \) is weakly lower — semicontinuous on \( H \), we obtain:

\[ J(u^*) \leq \lim \inf_{h \to 0^+} J(u_h) \leq \lim_{h \to 0^+} J(v_h) = J(u) . \]

The uniqueness of the solution of \( (\mathcal{P}) \) implies \( u = u^* \) and even the whole sequence \( \{u_h\} \) converges weakly to \( u \). Furthermore, we may write

\[ J(u_h) = J(u) + A(u, u_h - u) - f(u_h - u) + \frac{1}{2} A(u - u_h, u - u_h) . \]

Hence

\[ \gamma/2 |u - u_h|^2 \leq J(u_h) - J(u) + A(u, u - u_h) + f(u_h - u) \to 0 \quad \text{if } \quad h \to 0^+ . \]
Remark 1.5. If $\mathcal{C}_h \subset \mathcal{C}$ for all $h \in (0,1)$ then (1.8) can be replaced by the coerciveness of $J$ on $\mathcal{C}$, i.e. by the assumption
\[
\|v\| \to \infty, \quad v \in \mathcal{C} \Rightarrow J(v) \to +\infty.
\]

Remark 1.6. Let $\mathcal{H}$ be another Hilbert space with the norm $\|\cdot\|_{\mathcal{H}}$, $H \subset \mathcal{H}$ with completely continuous imbedding. Assume that a constant $c > 0$ exists such that
\[
(1.10) \quad \|v\|^2 \leq c(\|v\|^2_{\mathcal{H}} + |v|^2) \quad \forall v \in H
\]
(for example $H = H^1(\Omega)$, $\mathcal{H} = L^2(\Omega)$, $|v| = (\int_{\Omega} |\nabla v|^2 \, dx)^{1/2}$, $\Omega$ a domain with a continuous boundary). Let all assumptions of Theorem 1.2 be satisfied. Then
\[
\|u - u_h\| \to 0, \quad h \to 0^+.
\]

Proof. From Theorem 1.2 and its proof it follows
\[
|u - u_h| \to 0, \quad h \to 0^+
\]
and
\[
u_h \to u \quad \text{in} \quad H \quad \text{if} \quad h \to 0^+.
\]
Since the imbedding of $H$ into $\mathcal{H}$ is completely continuous, $u_h \to u$ in $\mathcal{H}$. The assertion is a consequence of (1.10).

2. APPROXIMATION OF CONTACT PROBLEMS BY FINITE ELEMENT METHOD

In this Section we describe the construction of finite-dimensional approximations of $\mathcal{H}$, $\mathcal{H}'$, i.e. of closed convex sets of admissible displacements in the problems $(\mathcal{P}_1)$, $(\mathcal{P}_2)$. For definitions see [1].

First we introduce some notations. Let $\Omega', \Omega''$ be two bounded disjoint domains with Lipschitz boundaries $\partial \Omega', \partial \Omega''$ and let us set $\Omega = \Omega' \cup \Omega''$. By $\mathcal{H}^k(\Omega)$, $k \geq 0$ integer, we denote the space, isomorphic with $[H^k(\Omega')]^2 \times [H^k(\Omega'')]^2$, i.e.
\[
u \in \mathcal{H}^k(\Omega) \iff u|_{\Omega'} \equiv u' \in [H^k(\Omega')]^2
\]
\[
u|_{\Omega''} \equiv u'' \in [H^k(\Omega'')]^2,
\]
where $H^k(\Omega^M)$, $M = ' , ''$ denote the Sobolev spaces. The norm and the seminorm in $\mathcal{H}^k(\Omega)$ is defined as follows:
\[
(2.1) \quad \|u\|^2_{k,\Omega} = \|u'\|^2_{k,\Omega'} + \|u''\|^2_{k,\Omega''},
\]
\[
|u|_{k,\Omega} = |u'|_{k,\Omega'} + |u''|_{k,\Omega''},
\]
where $\|u\|^2_{k,\Omega^M}$ and $|u|_{k,\Omega^M}$ ($M = ' , ''$) are the usual norms and seminorms, respectively in $[H^k(\Omega^M)]^2$. In what follows, we shall consider the problems $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ separately.
2.1. APPROXIMATION OF CONTACT PROBLEMS WITH A BOUNDED CONTACT ZONE

Let us consider the following decompositions of $\partial \Omega'$, $\partial \Omega''$ (see [1] — Section 1.):

$$
\partial \Omega' = \bar{\Gamma}_u \cup \bar{\Gamma}_t \cup \Gamma_K, \quad \partial \Omega'' = \bar{\Gamma}_0 \cup \bar{\Gamma}_t \cup \Gamma_K,
$$

where $\Gamma_u$, $\Gamma_K$ are non-empty parts of $\partial \Omega'$. Let us recall that

$$
V = \{ v \in \mathcal{H}^1(\Omega) \mid v = 0 \text{ on } \Gamma_u, \ v_n = 0 \text{ on } \Gamma_0 \} \quad \mathcal{H} = \{ v \in V \mid v_n + v_{n}'' \leq 0 \text{ on } \Gamma_K \}.
$$

A. First let us suppose that both $\Omega'$, $\Omega''$ are polygonal domains, $\Gamma_K = \sum_{i=1}^{m} \Gamma_{K,i}$, where $\bar{\Gamma}_{K,i}$ denotes a straight-line segment $A_i A_{i+1}$. Let $\mathcal{T}_h$ and $\mathcal{T}_h''$ be triangulations of $\Omega'$ and $\Omega''$, respectively, having common nodes on $\Gamma_K$ and such that $A_1, \ldots, A_m$ as well as all boundary points of $\Gamma_u$, $\Gamma_0$, $\Gamma_t$, $\Gamma_t'$, $\Gamma_K$ belong to nodes of $\mathcal{T}_h$ and $\mathcal{T}_h''$. We set $\mathcal{T}_h = \mathcal{T}_h' \cup \mathcal{T}_h''$. Let $h$ denote the maximal side and $\vartheta$ the minimal interior angle of all triangles $T_i \in \mathcal{T}_h$. Assume that a system of $\{ \mathcal{T}_h \}$, $h \to 0^+$ is regular, i.e. a constant $\alpha > 0$ exists such that $\vartheta \geq \alpha$ if $h \to 0^+$.

We define

$$
V_h = \{ v \in \left[ C(\bar{\Omega}') \right]^2 \times \left[ C(\bar{\Omega}'') \right]^2 \cap V \mid v \big|_T \in \left[ P_1(T) \right]^2 \text{ for all } T \in \mathcal{T}_h \},
$$

where $P_1(T)$ is the set of linear polynomials, defined on $T$.

Let $a_{j_i}$, $j = 1, \ldots, m_i$ be the nodes of $\mathcal{T}_h$, lying on $\bar{\Gamma}_{K,i}$ ($a_{i_1} \equiv A_i, a_{i_m} \equiv A_{i+1}$), $i = 1, \ldots, m$ and $n^i$ be the outward unit normal of the side $\Gamma_{K,i}$ related to $\partial \Omega'$. Let us define:

$$
\mathcal{K}_h = \{ v \in V_h \mid n^i \cdot (v' - v'') (a_{j_i}) \leq 0, \ i = 1, \ldots, m; \ j = 1, \ldots, m_i \}.
$$

It is readily seen that $\mathcal{K}_h$ are finite-dimensional approximations of $\mathcal{H}$. Moreover, it holds:

**Lemma 2.1.** $\mathcal{K}_h = \mathcal{K}$ for every $h \in (0, 1)$.

**Proof.** Let $v \in \mathcal{K}_h$. Then $n^i \cdot (v' - v'')|_{\Gamma_{K,i}}$ is piecewise linear function on $\bar{\Gamma}_{K,i}$. Hence

$$
n^i \cdot (v' - v'') \leq 0 \text{ on } \bar{\Gamma}_{K,i} \Rightarrow n^i \cdot (v' - v'') (a_{j_i}) \leq 0, \ j = 1, \ldots, m_i.
$$

B. Next we shall consider the case, when $\Omega'$, $\Omega''$ are domains with more general boundaries. For the sake of simplicity we restrict ourselves to the case, when only $\Gamma_K$ is curved. Let $\psi$ be a continuous concave (or convex) function, defined on $\langle a, b \rangle$ (see Fig. 1), the graph of which is $\bar{\Gamma}_K$. We choose $(m + 1)$ points $A_1, \ldots, A_{m+1}$ on $\Gamma_K$ in such a way that $A_1$ and $A_{m+1}$ are boundary points of $\bar{\Gamma}_K$. Let $A_i, A_{i+1} \in \bar{\Gamma}_K, S \in \Omega^M,$
By a curved element \( T \) we call a closed set bounded by the straight lines \( S A_i, S A_{i+1} \) and the arc \( A_iA_{i+1} \) such that \( T \subset \Omega^M, M = \cdot \), . The minimal interior angle of the triangle \( A_iA_{i+1}S \) is called the minimal angle of the curved element \( T \). A triangulation \( \mathcal{T}_h \) of \( \Omega = \Omega' \cup \Omega'' \) contains curved elements along \( \Gamma_K \) and internal triangular elements. By the symbols \( h \) and \( \theta \) we denote the maximal of diameter and the minimal interior angle, respectively, of all elements \( T \in \mathcal{T}_h \). Analogously as in the previous case we define a regular system of triangulations.

Define
\[
V_h = \{ v \in [C(\Omega')]^2 \times [C(\Omega'')]^2 \cap V | v|_T \in [P_1(T)]^2 \forall T \in \mathcal{T}_h \}
\]
\[
\mathcal{K}_h = \{ v \in V_h \ | \ n \cdot (v' - v^n) (A_i) \leq 0, \ i = 1, \ldots, m + 1 \}.
\]

It is easy to see that also in this case \( \mathcal{K}_h \) represents a finite - dimensional approximation of \( \mathcal{K} \), but \( \mathcal{K}_h \notin \mathcal{K} \), in general.

Let
\[
\mathcal{L}(v) = \frac{1}{2} \int_\Omega \tau_{ij}(v) e_{ij}(v) \, dx - \int_\Omega F \cdot v \, dx - \int_{\Gamma^+ \cap \Gamma^-} P \cdot v \, ds
\]
be the functional of the total potential energy, \( F \in L^0(\Omega), P \in [L^2(\Gamma_r)]^2 \times [L^2(\Gamma')]^2, \ e_{ij}(v) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i), \ \tau_{ij}(v) = c_{ijkl} e_{kl}(v); \) the coefficients \( c_{ijkl} \) are bounded and measurable in \( \Omega \),
\[
c_{ijkl} = c_{jikl} = c_{klij} \quad \text{a.e. in} \ \Omega
\]
and a positive constant \( c_0 \) exists such that
\[
(2.7) \quad c_{ijkl} e_{ij} e_{kl} \geq c_0 e_{ij} e_{ij} \quad \text{a.e. in} \ \Omega)
\]
holds for any symmetric \( e_{ij} \).

1) A repeated Latin index implies summation over the range 1,2.
An approximation of the contact problem with a bounded contact zone is defined as the solution of the following problem:

\[(\mathcal{P}_{1h}) \quad \text{find } u_h \in \mathcal{X}_h \text{ such that } \mathcal{L}(u_h) = \min_{v \in \mathcal{X}_h} L(v),\]

where \(\mathcal{X}_h\) is given by (2.3) or (2.5).

2.2. APPROXIMATION OF CONTACT PROBLEMS WITH AN ENLARGING CONTACT ZONE

Let \(\Omega', \Omega''\) be bounded domains with the following decomposition of the boundaries \(\partial \Omega', \partial \Omega''\):

\[\partial \Omega' = \Gamma_0 \cup \Gamma_i^' \cup \Gamma_K^' \quad \text{and} \quad \partial \Omega'' = \Gamma_0 \cup \Gamma_i^" \cup \Gamma_K^",\]

where

\[\Gamma_K^' = \{(\xi, \eta) \mid a \leq \eta \leq b, \, \xi = f'(\eta)\},\]

\[\Gamma_K^" = \{(\xi, \eta) \mid a \leq \eta \leq b, \, \xi = f''(\eta)\},\]

\(f', f''\) are continuous functions on \(\langle a, b \rangle\) (for details see [1]). The space \(V\) is defined as in the previous case. Define

\[\mathcal{X}_\varepsilon = \{v \in V \mid v_e - v_{e'} \leq \varepsilon, \forall \eta \in \langle a, b \rangle\},\]

where \(\varepsilon(\eta) = f'(\eta) - f''(\eta)\) is the distance of \(\Gamma_K^', \Gamma_K^"\) before the deformation and \(v_e, v_{e'}\) are projections of \(v', v''\) into the fixed direction \(\xi\).

For simplicity we restrict ourselves to the case when only \(\Gamma_K^', \Gamma_K^"\) are curved and the functions \(f', f''\), describing these arcs are twice continuously differentiable on \(\langle a, b \rangle\). Curved element \(T\) is defined in the same way as in the previous case-Part B. For the construction of finite-dimensional spaces on \(T\), we use the technique, developed in [3].

Let \(\hat{T}\) be the triangle with the vertices: \([0, 0], [1, 0], [0, 1]\). Let \(A_i, A_{i+1} \in \Gamma_K^', S \in \Omega'\) (for example), and let \(x = \varphi(s), y = \psi(s), s \in \langle 0, 1 \rangle\), \(\varphi, \psi \in C^2(\langle 0, 1 \rangle)\) be a parametric representation of the arc \(A_iA_{i+1}\) and \(T\) the curved element, determined by \(A_i, A_{i+1}, S\). Then we can construct the mapping \(F_T : R_2 \rightarrow R_2\), which is \(C^1\)-diffeomorphism \(\hat{T}\) onto \(T\). Let \(\hat{P} = P_1\) be a set of linear polynomials defined on \(\hat{T}\). Then we set

\[(2.8) \quad P(T) = \{p \mid \exists \hat{p} \in \hat{P} : p = \hat{p} \circ F_T^{-1}\},\]

where \(F_T^{-1}(T) = \hat{T}\).

The triangulation \(\mathcal{T}_h = \mathcal{T}'_h \cup \mathcal{T}''_h\) of \(\Omega\) consists of curved elements along \(\Gamma_K^', \Gamma_K^"\) and interior triangles.
Elements along \( \Gamma'_K \), \( \Gamma''_K \) are constructed in the following way: let \( \{C_j\}^m_{j=1} \) be a division of \( \langle a, b \rangle \), \( C_1 \equiv a \), \( C_m \equiv b \), \( A_j \), \( B_j \) being the intersections of perpendicular lines at \( C_j \) with \( \Gamma'_K \), \( \Gamma''_K \), respectively. Points \( A_j \) and \( B_j \) coincide with the nodes of \( \mathcal{T}_h \) on \( \Gamma'_K \) and \( \Gamma''_K \), respectively (cf. Fig. 2).

![Fig. 2.](image)

Let us define

\[
V_h = \{ v \in V | v \mid_T \in [P(T)]^2 \quad \forall T \in \mathcal{T}_h \},
\]

where \( P(T) = P_1(T) \) if \( T \) is a triangle or \( P(T) \) is defined by (2.8) if \( T \) is a curved element. Let

\[
\mathcal{K}_h = \{ v \in V_h \mid v(B_j) - v(A_j) \leq \delta(C_j), \quad j = 1, \ldots, m \}.
\]

It is easy to see that \( \mathcal{K}_h \) is the finite-dimensional approximation of \( \mathcal{K}_e \) and \( \mathcal{K}_h \neq \mathcal{K}_e \) in general.

An approximation of the contact problem with an enlarging contact zone is defined as the solution of the following problem:

\[
(\mathcal{P}_{2h}) \quad \text{find } u_h \in \mathcal{K}_h \text{ such that } \quad \mathcal{L}(u_h) = \min_{v \in \mathcal{K}_h} \mathcal{L}(v)
\]

3. ERROR ESTIMATES

In this Section we establish the rate of convergence of approximate solutions \( u_h \) (defined in the previous Section), provided the exact solution is smooth enough. We shall analyze the problems \( (\mathcal{P}_1) \) and \( (\mathcal{P}_2) \) separately. The results of Section 1 will be used with

\[
(3.1) \quad A(u, v) = \int_{\Omega} \tau_{ij}(u) \epsilon_{ij}(v) \, dx,
\]
First we recall the well-known Green's formula. Let $Q \subset \mathbb{R}^2$ be a domain with a Lipschitz boundary $\partial Q$. Let us introduce

$$Y(Q) = \{ \tau \in [L^2(Q)]^4 \mid \tau_{ij} = \tau_{ji} \text{ a.e. in } Q \}$$
$$\hat{Y}(Q) = \{ \tau \in Y(Q) \mid \tau_{ij,j} \in L^2(Q), \quad i = 1, 2 \},$$

where $\tau_{ij,j} = \partial \tau_{ij}/\partial x_j$ is taken in the distribution sense. Then there exists a unique $T \in \mathcal{L}(Y(Q), (H^{-1/2}(\partial Q))^2)$ such that

$$\int_{\Omega} \tau_{ij} e_{ij}(v) \, dx = - \int_{\Omega} \tau_{ij,j} v_i \, dx + \langle T, v \rangle \quad \forall \tau \in \hat{Y}(Q), \quad \forall v \in [H^1(Q)]^2,$$

where $\langle , \rangle$ denotes the duality between $[H^{-1/2}(\partial Q)]^2$ and $[H^{1/2}(\partial Q)]^2$. If $T \in [L^2(\partial Q)]^2$ then

$$\langle T, v \rangle = \int_{\partial \Omega^M} T_i v_i \, ds.$$

Using (3.3) with $Q = \Omega'$ and $Q = \Omega''$, respectively, we obtain

$$\int_{\Omega} \tau_{ij} e_{ij}(v) \, dx = - \int_{\Omega} \tau_{ij,j} v_i \, dx + \langle T', v' \rangle_{\partial \Omega'} + \langle T'', v'' \rangle_{\partial \Omega''} \quad \forall \tau \in \hat{Y}(\Omega), \quad \forall v \in \mathcal{H}^1(\Omega),$$

where $T_{M} \in [H^{-1/2}(\partial \Omega')]^2$ and $\langle , \rangle_{\partial \Omega}$ denotes the duality between $[H^{-1/2}(\partial \Omega')]^2$ and $[H^{1/2}(\partial \Omega')]^2$. Henceforth we assume for simplicity that $T' \in [L^2(\partial \Omega')]^2$, $T'' \in [L^2(\partial \Omega'')]^2$. Finally, let us denote

$$|v|^2 = \int_{\Omega} e_{ij}(v) e_{ij}(v) \, dx.$$

### 3.1. CONTACT PROBLEMS WITH A BOUNDED CONTACT ZONE

Let the weak solution $u$ of the problem $(\mathcal{P}_1)$ be such that $\tau^M(u) \in \hat{Y}(\Omega^M), \quad M = ' ', ''$. Then using the definition of $(\mathcal{P}_1)$ and (3.4) we obtain (cf. [1] — Th. 1.1):

\begin{align*}
(3.5) & \quad \tau_{ij,i}(u) + F_i = 0 \quad \text{in } \Omega, \quad i = 1, 2; \\
(3.6) & \quad u = 0 \quad \text{on } \Gamma_a; \\
(3.7) & \quad \tau_{ij}(u) n_j = P_i \quad \text{on } \Gamma_\tau \equiv \Gamma' \cup \Gamma'', \quad i = 1, 2; \\
(3.8) & \quad u_a = 0, \quad T_i(u) = 0 \quad \text{on } \Gamma_0; \\
\end{align*}

1) $H^{1/2}(\partial Q)$ is the space of traces of all functions belonging to $H^1(Q)$, $H^{-1/2}(\partial Q)$ denotes the dual space to $H^{1/2}(\partial Q)$. We shall write simply $\langle T, v \rangle$ instead of $\langle T(\tau), v \rangle$. 

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\( u'_n + u''_n \leq 0 \) on \( \Gamma_K \),

\( T_n'(u') = T_n''(u'') = T_n(u) \leq 0 \) on \( \Gamma_K \),

\( T_i'(u') = T_i''(u'') = 0 \) on \( \Gamma_K \),

\( T_n(u) \cdot (u'_n + u''_n) = 0 \) on \( \Gamma_K \),

where \( T_n(u) \) and \( T_i(u) \) are the normal and tangential components of the stress vector, respectively.

A. Let us suppose that \( \Omega', \Omega'' \) are polygonal domains, \( \mathcal{K}_h \) is defined by (2.3).

**Theorem 3.1.** Let there exist solutions \( u_h, u_{h_0} \) of \((\mathcal{P}_1), (\mathcal{P}_1h)\), respectively, such that \( u \in H^2(\Omega) \cap \mathcal{K}, \tau_M(u) \in \mathcal{Y}(\Omega_M), \ M = ' , '', u', u'' \in [W^{1,\infty}(\Gamma_K), i = 1, ..., m. \)

Moreover, let us suppose that the number of points on \( \Gamma_K \), where the contact changes from binding to nonbinding, is finite. Then

\( |u - u_h| \leq ch\left[|u|_{2,\Omega}^2 \sum_{i=1}^m \left| T_n(u) \right|_{x, \Gamma_K,i} \left( |u'_i|_{1, \infty, \Gamma_K,i} + |u''_i|_{1, \infty, \Gamma_K,i} \right) \right]^{1/2} \)

if the system \( \{\mathcal{T}_h\}, h \to 0+ \) is regular.

**Proof.** From (1.1), Remarks 1.1, 1.3 and Lemma 2.1 we obtain

\( A(u - u_h, u - u_h) \leq f(u - v_h) + \frac{1}{2} A(u - v_h, u - v_h) + A(u, v_h - u) \)

\( \forall v_h \in \mathcal{K}_h. \)

The definition of \((\mathcal{P}_1)\) and the Green's formula, (3.5)-(3.8), (3.10) and (3.11) yields:

\( f(u - v_h) + A(u, v_h - u) = \int_{\Gamma_K} T_n(u) \{ (v'_n - u'_n) - (v''_n - u''_n) \} \, ds \)

\( \forall v_h \in \mathcal{K}_h. \)

(In the following the normal \( n \) will be related to \( \Omega' \) only.) To prove (3.13) we make a special choice of \( v_h \), namely \( v_h = u_I \), where \( u_I \) is a piecewise linear Lagrange interpolate of \( u \) over the triangulation \( \mathcal{T}_h. \) It is readily seen that \( u_I \in \mathcal{K}_h. \) In fact,

\( n^i \cdot (u'_I - u''_I) \cdot (a_j^I) = n^i \cdot (u' - u'') (a_j^I) \leq 0. \)

Using the well - known approximative properties of \( u_I \), we deduce

\( |A(u_I - u, u_I - u)| \leq c|u - u_I|^2 \leq ch^2|u|^2_{2,\Omega} \)

Let

\( \Gamma_{K,i}^{-} = \{ x \in \Gamma_{K,i} | (u'_n - u''_n)(x) < 0 \} \)

\( \Gamma_{K,i}^{0} = \{ x \in \Gamma_{K,i} | (u'_n - u''_n)(x) < 0 \} \).
Denoting $\mathcal{U}_I \equiv (u_I - u^n). n^i = [(u' - u^n). n^i]_I$, we have

\begin{equation}
\mathcal{U}_I \equiv 0
\end{equation}

(3.17)

on every side $s^i_j = a^i_j a^j_{i+1} \subseteq \Gamma^0_{K,i}$. Moreover, $T_n(u) = 0$ a.e.on $\Gamma_{K,i}$ follows from (3.12). From this and (3.17) we obtain

$$f(u - u_I) + A(u, u_I - u) = \sum_{i=1}^m \sum_{s^i_j \in \mathcal{S}_K} T_n(u) \left[ (u'_n - u^n)_I - (u_n' - u^n) \right] ds ,$$

where $\mathcal{S}_K$ is the system of all $s^i_j \subseteq \Gamma_{K,i}$ containing both points of $\Gamma^0_{K,i}$ and $\Gamma^-_{K,i}$. Using the assumptions of the Theorem, we have

$$\int_{s^i_j} T_n(u) \left[ \mathcal{U}_I - (u'_n - u^n) \right] ds \leq \frac{h}{h} \left\| T_n(u) \right\|_{\infty, s^i_j} \cdot \frac{\left\| \mathcal{U}_I - (u'_n - u^n) \right\|_{\infty, s^i_j} \leq \frac{ch^2}{h} \left\| T_n(u) \right\|_{\infty, s^i_j} \left| u'_n - u^n \right|_{1, \infty, s^i_j} \leq \frac{ch^2}{h} \left\| T_n(u) \right\|_{\infty, s^i_j} \cdot \left( \left| u' \right|_{1, \infty, s^i_j} + \left| u'' \right|_{1, \infty, s^i_j} \right) .$$

Due to the assumptions, the number of all $s^i_j \in \mathcal{S}_K$, $i = 1, \ldots, m$ can be bounded from above independently of $h$. Using (3.14) and (3.16), the assertion (3.13) follows.

Remark 3.1. Some sufficient conditions for the existence and uniqueness of solution of the problem $(\mathcal{P}_I)$ are given in [1]. Since $\mathcal{K}_h \subset \mathcal{K}$, the same conditions are true for the problem $(\mathcal{P}_{1h})$. The previous Theorem 3.1 however, doesn’t require the uniqueness of the solutions.

Remark 3.2. The same rate of convergence $O(h)$ can be obtained for example if $u \in \mathcal{H}^2(\Omega) \cap \mathcal{K}$, $\tau^M(u) \in \overline{\mathcal{Y}}(\Omega^M)$, $M = \tau$, " and $u'_n, u''_n \in H^2(\Gamma_{K,i}) \forall i = 1, \ldots, m$. Under the single assumptions $u \in \mathcal{H}^2(\Omega) \cap \mathcal{K}$, $\tau^M(u) \in \overline{\mathcal{Y}}(\Omega^M)$, $M = \tau$, " we obtain

$$\left| u - u_h \right| = O(h^{3/4}) \quad h \to 0+ .$$

Remark 3.3. Let on $\Gamma_{K_0} \subset \Gamma_K$ (given a priori) $\Omega'$ and $\Omega''$ are in a bilateral contact i.e. $u'_n + u''_n = 0$ on $\Gamma_{K_0}$ and let $\Gamma_{K_0}$ be such that the rigid virtual displacements, satisfying the bilateral contact, reduce to zero field, i.e.

$$v \in \mathcal{K} \cap V, \quad v'_n + v''_n = 0 \quad \text{on} \quad \Gamma_{K_0} \Leftrightarrow v \equiv 0 .$$

Then using the inequality of Korn’s type, we obtain the rate of convergence in $\mathcal{H}^1(\Omega) - \text{norm}$.

In the above error estimates we needed very strong regularity assumptions, concerning the solution $u$. Unfortunately, there are no reasons to expect such a great smoothness in a general case. This is why we are going to prove the convergence of $u_h$ to $u$ without estimating the rate of convergence, using no regularity hypothesis. To this end we need the following.

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Lemma 3.1. Let us suppose that $\Gamma_K \cap \Gamma_u = 0, \Gamma_K \cap \Gamma_0 = 0$ and there exists only a finite number of boundary points $\bar{\Gamma}_f \cap \Gamma_K, \bar{\Gamma}_u \cap \bar{\Gamma}_f, \bar{\Gamma}_r \cap \bar{\Gamma}_0$. Then the set

$$\mathcal{M} = \mathcal{H} \cap [C^\infty(\bar{\Omega})]^2 \times [C^\infty(\bar{\Omega}')]^2$$

is dense in $\mathcal{H}$ in $\mathcal{H}^1(\Omega) - \text{norm}$.

Proof. Let $u \in \mathcal{H}$ be a fixed arbitrary function. Consider a system of open domains $\{B_i\}_{i=0}^r$, which cover $\Omega' \cup \Omega''$, such that: $\bar{B}_0 \subset \Omega', \bar{B}_1 \subset \Omega''$,

$$\Gamma_K \subset \bigcup_{j=2}^k B_j, \quad (k < r)$$

$$\Gamma_K \cap B_i \neq 0 \iff 2 \leq i \leq k .$$

We say that a point $P \in \partial \Omega' \cup \partial \Omega''$ is a singular point, if $P$ is either a vertex of the polygonal boundary or a point of

$$\Gamma_K \cap \bar{\Gamma}_f, \quad \bar{\Gamma}_u \cap \bar{\Gamma}_f \quad \text{or} \quad \bar{\Gamma}_0 \cap \bar{\Gamma}_r .$$

We may assume that each $B_j$ contains at most one singular point. Denote $\{\varphi_i\}, i = 0, 1, \ldots, r$, the corresponding decomposition of unity (i.e. $\varphi_i \in C^\infty(\bar{B}_i)$, $0 \leq \varphi_i \leq 1$, $\sum_{i=0}^r \varphi_i(x) = 1$ $\forall x \in \bar{\Omega}' \cup \bar{\Omega}''$). Introducing

$$u^j = u \varphi_j, \quad j = 0, 1, \ldots, r ,$$

we have $\text{supp } u^j \subset B_j$, $u^j \in \mathcal{H}^1(\Omega)$, $\sum_{j=0}^r u^j = u$. For each $u^j$ we shall construct infinitely differentiable and close functions satisfying the boundary conditions. To this end we divide the system $\{B_j\}$ into several groups.

1. group. Let $j \leq k$ and $\{B_j\}$ do not contain any singular point. Introduce local Cartesian coordinates $(\xi, \eta)$, where $\xi$-axis coincides with the tangent and $\eta$-axis with the normal $\eta$' with respect to $\Gamma_K$. Then (omitting the indeces $j$), we have

$$\Gamma_K \cap B = \{ (\xi, \eta) \mid |\xi| < \xi_0, \eta = 0 \} ,$$

$$u^M = u_\xi^e \xi + u_\eta^e \eta , \quad M = ' , $'$ ,$$

$$(\epsilon_\xi, e_\eta \text{ are unit basis vectors}),$$

$$u_\eta' + u_\eta'' = u_\eta' - u_\eta'' \leq 0 \quad \text{on } \Gamma_K .$$

Let us extend $u_\eta'$ into $B \cap \Omega''$ and $u_\eta''$ into $B \cap \Omega'$ in such a way that the extension is even with respect to $\eta$. Regularizing the extended function $Eu_\eta'$ by means of the kernel $\omega(x, x')$, we obtain

$$R_x E u'_\eta(x) = \int_B \omega(x - x') E u_\eta'(x') \, dx' , \quad x' = (\xi', \eta') .$$

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There exists a function \( v \in H^1(B) \) such that
\[
v \leq 0 \quad \text{in} \quad B, \quad \text{supp} \ v \subset B, \quad v = u'_\eta - u''_\eta \leq 0 \quad \text{on} \quad \Gamma_K,
\]
(see e.g. [7] — chpt. 2, the proof of Th. 5.7). Then
\[
(3.20) \quad Eu'_\eta - Eu''_\eta = v + z,
\]
where \( z \in H^1(B) \), the restriction \( z|_{\partial M} \in H^1_0(B \cap \Omega^M) \), \( M = ' ', '' \).
Obviously, it holds
\[
R_xv \leq 0 \quad \text{on} \quad \Gamma_K, \quad R_x v \in C^\infty_0(B), \quad R_xv \to v \quad \text{in} \quad H^1(B) \quad \text{for} \quad \varepsilon \to 0
\]
and approximations \( z^M_\varepsilon \in C^\infty_0(B \cap \Omega^M) \) exist such that \( z^M_\varepsilon \to z|_{\partial M} \) in \( H^1(B \cap \Omega^M) \).
Consequently, we have
\[
(3.21) \quad R_xv + z_\varepsilon \leq 0 \quad \text{on} \quad \Gamma_K, \quad \text{setting} \quad z_\varepsilon = (z'_\varepsilon, z''_\varepsilon) \in C^\infty_0(B),
\]
\[
(3.22) \quad u^M_\eta \to u^M_\eta \quad \text{in} \quad H^1(B \cap \Omega^M) \quad \text{for} \quad \varepsilon \to 0.
\]
Moreover, we have
\[
(3.23) \quad u'_\eta - u''_\eta = R_xv + z_\varepsilon \leq 0 \quad \text{on} \quad \Gamma_K.
\]
The components \( u^M_\eta \) can be regularized by an arbitrary way.

2. group. Let \( j \leq k \) and \( B_j \) contain a vertex \( P \in \Gamma_K \). We use a skew coordinate basis for the components of \( u \). Let \( e^1, e^2 \) be unit tangential and \( n^1, n^2 \) unit outward
normal vectors, with respect to $\partial \Omega'$ — see Fig. 3. Then we may write (omitting indeces $j$)

$$u' = \sum_{p=1}^{2} \frac{u'(p)}{e^p \cdot n^p} e^p,$$

where $u'(p) = u' \cdot n^p$, i.e.

$$u'(p) = u'_n \text{ on } \Gamma^{(p)}, \; p = 1, 2.$$

The same decomposition is valid for $u''$ and

$$u''(p) = - u''_n \text{ on } \Gamma^{(p)}, \; p = 1, 2.$$

Altogether we have

$$u'_n + u''_n = u'(p) - u''(p) \leq 0 \text{ on } \Gamma^{(p)}.$$

First let us consider the components $u^{M(2)}$. We map the angular domain $B \cap \Omega'$ into the upper halfplane $\{(\xi, \eta) \mid \eta > 0\}$ by means of a Lipschitz mapping $T$ such that $TT^{(2)}$ is the positive and $TT^{(1)}$ the negative $\xi$-axis. Let us extend the functions

$$\hat{u}^{M(2)}(\xi, \eta) = u^{M(2)}(T^{-1}(\xi, \eta))$$

across the $\xi$-axis to get functions even in $\xi$. Set

$$E u^{M(2)}(x) = E \hat{u}^{M(2)}(T(x)).$$

Regularizing, we obtain

$$R_x E u^{(2)}(x) \in C^0(B).$$

Let us extend the function

$$\hat{u} = u^{(2)} - \hat{u}^{(2)} \leq 0 \text{ on } TT^{(2)}$$

from the positive onto the negative $\xi$-axis in such a way that the extension $E \hat{u}$ is even in $\xi$. Then a function $\hat{v} \in H^1(TB)$ exists such that $\hat{v} = E \hat{u}$ on the $\xi$-axis, $\hat{v} \leq 0$ in $TB$, supp $\hat{v} \subset TB$. Defining

$$v(x) = \hat{v}(Tx),$$

then $v \in H^1(B)$, supp $v \subset B$, $v \leq 0$ in $B$, $v = u^{(2)} - u''^{(2)}$ on $\Gamma^{(2)}$.

Consequently, we may write

$$Eu^{(2)} - Eu^{(2)} = v + z,$$

where

$$z \in H^1(B), \; \text{supp } z \subset B, \; z = 0 \text{ on } \Gamma^{(2)}.$$

Regularizing, we obtain

$$R_x v \leq 0 \text{ on } \Gamma, \; R_x v \to v \text{ in } H^1(B), \; x \to 0.$$

There exists a function $w \in H^1(B)$, supp $w \subset B$, such that $w = z$ on $\Gamma^{(1)} \cup \Gamma^{(2)}$ and $w = 0$ in an “angular neighbourhood” $|\theta| < \theta_0$ of the part $\Gamma^{(2)}$. Defining a “shifted”
function 
\[ w_\lambda(x) = w(x + \lambda e^1), \quad \lambda \in \mathbb{R}^1, \quad \lambda > 0, \]
then for \( x < C\lambda \) it holds \( R_\lambda w_\lambda = 0 \) on \( \Gamma^{(2)} \),
\[
\| R_\lambda w_\lambda - w \| \leq \| R_\lambda w_\lambda - w_\lambda \| + \| w_\lambda - w \| \rightarrow 0 \quad \text{for} \quad \lambda \rightarrow 0,
\]
with the norms in \( H^1(B) \).

Furthermore, since we have
\[
z = w + z_0, \quad z_0|_{\Omega M} \in H_0^1(B \cap \Omega^M),
\]
\[
(3.26) \quad R_\lambda w_\lambda + z_0 = 0 \quad \text{on} \quad \Gamma^{(2)}, \quad z_0^M \in C_0^\infty(B \cap \Omega^M),
\]
\[
R_\lambda w_\lambda + z_0 \rightarrow w + z_0 = z \quad \text{in} \quad H^1(B).
\]

Let us set \( u''^{(2)} = R_\lambda E u''^{(2)} |_{\Omega'} \)
\[
(3.27) \quad u''^{(2)} = [R_\lambda E u''^{(2)} - (R_\lambda w_\lambda + z_0)] |_{\Omega'}
\]
Then we obtain on the basis of (3.24), (3.25), (3.26) that
\[
(3.27) \quad u''^{(2)} \rightarrow u''^{(2)} \quad \text{in} \quad H^1(\Omega^M \cap B)
\]
and
\[
(3.27) \quad u''^{(2)} - u''^{(2)} = R_\lambda v + R_\lambda w_\lambda + z_0 \leq 0 \quad \text{on} \quad \Gamma^{(2)}.
\]

The component \( u''^{(2)} \) can be treated in a parallel way. Since the Cartesian components \( w_\lambda \) of an arbitrary vector \( w \) can be written as
\[
w_k = a_1 w^{(1)} + a_2 w^{(2)}
\]
with fixed constants \( a_1, a_2, \) it holds
\[
\| w_k \|^2 \leq C \sum_{p=1}^2 \| w^{(p)} \|^2, \quad k = 1, 2.
\]

![Fig. 4.](image-url)
Hence it follows that defining
\[ u^M_x = \sum_{p=1}^{2} u_{x}^{M(p)} e^p |e^p| n_p, \quad M = ' , '' , \]
the relations (3.27) imply
\[ \| u^j - u^l \|_{1, \Omega} \to 0, \quad \text{for} \quad \varepsilon \to 0, \quad \lambda \to 0, \quad \varepsilon < C\lambda . \]

3. group. Let \( j \leq k \) and let \( B_j \) contain a point \( P \in \Gamma_K \cap \Gamma_r \). We place the cartesian system into the point \( P \) so that \( x_1 \)-axis coincides with \( \Gamma_K \). (see Fig. 4). Then we have
\[ u_n' + u_n'' = -u_2' + u_2'' \leq 0 \quad \text{on} \quad \Gamma_K . \]

We can proceed in a way analogous to that of the 2. group, replacing the components \( u^{M(2)} \) by \( u_2^M \) and the ray \( \Gamma^{(2)} \) by \( \Gamma_K \).

4. group. Let \( B_j \) contain a point \( P \in \Gamma_0 \cap \Gamma_r \), which can also be a vertex. If the cartesian local system is such that \( x_1 \)-axis coincides with \( \Gamma_0 \), then \( u_n'' = \pm u_2'' = 0 \) holds on \( \Gamma_0 \). There exists a function \( v \in H^1(B_j \cap \Omega) \) such that \( \text{supp} \ v \subset B_j \), \( v = u_2'' \) on \( \partial \Omega'' \), \( v = 0 \) in an angular neighbourhood of the ray \( \Gamma_0 \) (with the vertex at \( P \)).

If we shift \( v \) to get \( v_\lambda(x) = v(x + \lambda) \) with a suitably chosen vector \( \lambda \), then \( R_\lambda v = 0 \) on \( \Gamma_0 \), \( R_\lambda v \to v \) in \( H^1(B_j) \) for \( \varepsilon < C|\lambda|, |\lambda| \to 0 \). Since \( u_2'' = v + z \), where \( z \in H^1_0 (B_j \cap \Omega) \), we have \( u_2'' = 0 \) on \( \Gamma_0 \) and
\[ u_2'' \equiv R_\lambda v + z \to u_2'' \quad \text{in} \quad H^1(B_j \cap \Omega) . \]

5. group. Let \( B_j \) contain a point \( P \in \Gamma_0 \cap \Gamma_r \). The approach, used for \( u_2'' \) in the 4. group, can be applied to both components \( u'_k, k = 1, 2 \).

The cases of \( B_j \cap \partial \Omega' \subset \Gamma_w \), \( B_j \cap \partial \Omega'' \subset \Gamma_0 \) and \( B_j \cap \partial \Omega^M \subset \Gamma_r \), as well as of \( B_0, B_1 \) are easy.

Finally, defining
\[ u^M_x = \sum_{j=0}^{r} u_x^{Mj} , \quad M = ' , '' , \]
we obtain
\[ \| u - u_h \|_{1, \Omega} \to 0 \quad \text{for} \quad \varepsilon \to 0, \quad \lambda \to 0, \quad \varepsilon < C|\lambda| . \]

Moreover, the functions \( u_x \) are infinitely differentiable and satisfy all the boundary condition involved in the definition of \( K \).

**Theorem 3.2.** Let \( \mathcal{L} \) be coercive on \( \mathcal{K} \) and let \( (\mathcal{P}_1) \) have precisely one solution. Let the assumptions of Lemma 3.1 be satisfied. Then for any regular system of triangulations \( \{ \mathcal{T}_h \} \) we have
\[ \| u - u_h \|_{1, \Omega} \to 0, \quad h \to 0+ . \]

**Proof.** It follows from Theorem 1.2, Remarks 1.4, 1.5 and 1.6. It remains to verify (1.6). Since the Lagrange interpolate \( v_\lambda \) of any \( v \in \mathcal{W} \) belongs to \( \mathcal{K}_h \) and \( \mathcal{W} \) is dense in \( \mathcal{K} \), the condition (1.6) is true.
B. Assume that $\Gamma_k$ is a curved arc. Let $K_h$ be defined by means of (2.5). The main difficulty consists in the fact that $K_h \notin K$, in general. First we present several auxiliary results.

**Lemma 3.2.** Let $Q \subset R_2$ be a bounded convex domain, the boundary of which is twice continuously differentiable and let $\{T_h\}, h \to 0+$ be a strongly $\alpha-\beta$-regular system of triangulations,\(^1\) where $\beta = 2$ and the maximal straight side in $T_h$ is not greater than the maximal chord of $\partial Q$. Then

$$\|u - u_h\|_{0, Q} \leq c h^{3/2} \|u\|_{2, Q}, \quad \forall u \in H^2(Q),$$

where $u_h$ denotes the piecewise linear Lagrange interpolant of $u$, $c > 0$ is independent of $u, h$.

**Proof.** — see [5].

**Lemma 3.3.** Let $p$ be a linear function, defined on the interval $(a, b)$. Then

$$\|p\|_{1, \langle a, b \rangle} \leq c (b - a)^{-1/2} \|p\|_{1/2, \langle a, b \rangle}.$$

**Lemma 3.4.** Assume that the arc $A(A_{i+1}$ is the curved side of a curved triangular element. Let $v \in P_1(T)$ and $T_h$ be the triangle generated by replacing the curved side by its chord. Then

$$\|v\|_{1, A(T, T_h)} \leq c h \|v\|_{1, T_h},$$

where $A(T, T_h) = (T - T_h) \cup (T_h - T)$, $c > 0$ is independent of $h$ and of $v$.

**Proof.** — see [4], p. 199.

**Theorem 3.3.** Let the problems $(P_\lambda)$ and $(P_{\lambda h})$ have solutions $u$ and $u_h$, respectively. Let $u \in H^2(\Omega) \cap K$, $\tau(u^M) \in Y(\Omega^M)$, $M = 1, \ldots, T(u) \in L^2(\Gamma_k)$ and let the norms $\|u\|_{1, \Omega}$ remain bounded. Assume that the system $\{T_h\}, h \to 0+$ is strongly $\alpha-\beta$-regular with $\beta = 2$ and the maximal straight side in $T_h$ is not greater than the maximal chord. Let the function $\psi$, describing $\Gamma_K$, be three times continuously differentiable. Then

$$\|u - u_h\| \leq c(u) h^{3/4}, \quad h \to 0+.$$

**Proof.** Using the definition of $(P_\lambda)$, (1.1), Remark 1.3 and the Green's formula, we deduce

$$\frac{1}{2} A(u - u_h, u - u) \leq \frac{1}{2} A(v_h - u; v_h - u) + \int_{\Gamma_k} T(u) [(v_n - u_n) - (v^w_n - u^w_n)] ds + \int_{\Gamma_k} T(u) [(v^W_h - u^W_h) - (v^W_n - u^W_n)] ds,$$

$$\forall v_h \in K_h, \quad \forall v \in K.$$

\(^1\) I.e. regular system, such that the ratio of any straight sides (or chords) in $T_h$ is bounded above by some constant $\beta$, independent of $h$. 279
Let \( v_h = u_I \), where \( u_I \) is the piecewise linear Lagrange interpolate of \( u \) on \( \Omega \). It is easy to see that \( u_I \in \mathcal{X}_h \) and we obtain

\[
A(u_I - u; u_I - u) \leq M \| u_I - u \|_{0,\Omega}^2 \leq ch^2 \| u \|_{2,\Omega}^2,
\]

\[
\int_{\Gamma_K} T_n(u) \left[ (u_I' - u') \cdot n - (u'' - u'') \cdot n \right] \, ds \leq c \left( \| u_I' - u' \|_{0,\Omega} + \| u'' - u'' \|_{0,\Omega} \right) \leq ch^{3/2} \| u \|_{2,\Omega},
\]

as follows from (3.28).

The most difficult is to estimate the term

\[
\int_{\Gamma_K} T_n(u) \left[ (v_n' - v_n') - (u_h' - u_{hn}) \right] \, ds, \quad v \in \mathcal{X}.
\]

In what follows we shall construct a function \( v \in \mathcal{X} \) such that (3.32) is small.

We identify the origin of the coordinate system \((x_1, x_2)\) with the point \( A_i \) (see Fig. 5).

Let \( \Sigma_i \) be a closed set bounded with the arc \( \Gamma_i \equiv s_i \subset \Gamma_K \), connecting \( A_i \) with \( A_{i+1} \), and the chord \( A_i A_{i+1} \). Let \( x \in \Sigma_i \). By the symbol \( P(x) \) and \( Q(x) \), respectively, we denote the intersection of the perpendicular line through the point \( x \) with \( s_i \) and \( A_i A_{i+1} \), respectively. Let \( T_i \subset \Omega' \), \( T_i'' \subset \Omega'' \), \( \partial T_i' \cap \partial T_i'' = s_i \) be the two adjacent curved elements. We extend any function \( v \in [P_1(T_i'')]^2 \) on \( T_i'' \cup \Sigma_i \) as follows:

\[
Ev \in [P_1(T_i'' \cup \Sigma_i)]^2 ;
\]

\[
Ev \big|_{T_i''} = v.
\]

In order to apply Lemma 3.2, we can proceed as follows. Let \( \bar{\Omega}' \) be a convex set with twice continuously differentiable boundary \( \partial \bar{\Omega} \subset \Gamma_K \). Let \( Eu \in H^2(\bar{E}_2) \) denotes the continuous Calderon extension (see [7]) of \( u' \in H^2(\Omega') \). Then, according to Lemma 3.2

\[
\| u' - u_i \|_{0,\Omega} \leq \| Eu' - (Eu_i') \|_{0,\bar{\Omega}'} \leq ch^{3/2} \| Eu' \|_{2,\bar{\Omega}'} \leq ch^{3/2} \| u' \|_{2,\Omega}.
\]

Analogously one can estimate \( \| u'' - u_i'' \|_{0,\Omega} \).

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For simplicity of notations, we use again the symbol $v$ instead of $E v$. Let us define functions $\mathcal{U}_h \ (i, l)$ again on $\bigcup \Sigma_i$ by means of the following relations:

$$
\mathcal{U}_h(x) = (u_h'(x) - u_\ast'(x)) \cdot n(P(x))
$$

$$
\tilde{U}_h(x) = (u_h'(Q(x)) - u_\ast'(Q(x))) \cdot n(P(x)) = (\tilde{u}_h' - \tilde{u}_\ast')(x) \cdot n(P(x)) ,
$$

where

$$
\tilde{u}_h(x) = u_h'(Q(x)), \quad \tilde{u}_\ast(x) = u_\ast'(Q(x)), \quad x \in \Sigma_i .
$$

Clearly

$$
\mathcal{U}_h(x) = \tilde{U}_h(x), \quad x \in \bigcup_{i=1}^m A_i A_{i+1}^1.
$$

Let $\Phi_i(x), \quad x \in A_i A_{i+1}$ be the linear Lagrange interpolates of $\mathcal{U}_h$ on $A_i A_{i+1}$ and let us define the function $\Phi$ on $\bigcup \Sigma_i$ as follows:

$$
\Phi(x) = \Phi_i(Q(x)), \quad x \in \Sigma_i , \quad i = 1, \ldots, m .
$$

It is readily seen that $\Phi \leq 0$ on $\Gamma_K$. Let us estimate $\| \Phi - \mathcal{U}_h \|_{0, R_K}$. We may write:

$$
(3.33) \quad \| \Phi - \mathcal{U}_h \|_{0, R_K} \leq \| \Phi - \tilde{U}_h \|_{0, R_K} + \| \tilde{U}_h - \mathcal{U}_h \|_{0, R_K} ,
$$

$$
\| \tilde{U}_h - \mathcal{U}_h \|_{0, R_K}^2 = \sum_{i=1}^m \| \tilde{U}_h - \mathcal{U}_h \|_{0, s_i}^2 \leq
$$

$$
\leq 2 \left( \sum_{i=1}^m \| u_h'(x) - \tilde{u}_h'(x) \|_{0, s_i}^2 + \sum_{i=1}^m \| u_\ast'(x) - \tilde{u}_\ast'(x) \|_{0, s_i}^2 \right) .
$$

Let $\tau$ be the arcs' parameter of the point $P(x) = (P_1(x), P_2(x))$ and denote $Q_1(x) = x_1$. Then for $M = ' , "$ we have

$$
u_{h_j} - \tilde{u}_{h_j} = \int_0^{P_1(x)} \frac{\partial}{\partial x_2} (u_{h_j} - \tilde{u}_{h_j}) \, dx_2 = \int_0^{P_2(x)} \frac{\partial}{\partial x_2} u_{h_j} \, dx_2 , \quad j = 1, 2 .
$$

Integrating and using Fubini's theorem we obtain

$$
\| u_{h_j} - \tilde{u}_{h_j} \|_{0, s_i}^2 \leq ch^2 \| u_{h_j} \|_{1, s_i}^2 , \quad j = 1, 2 .
$$

From this and Lemma 3.4 we have

$$
(3.34) \quad \| \tilde{U}_h \|_{0, R_K} \leq ch^2 \left( \sum_{i=1}^m \| u_h \|_{1, s_i} + \sum_{i=1}^m \| u_\ast \|_{1, s_i} \right) \leq ch^3 \| u_h \|_{1, s} .
$$

Let us estimate $\| \Phi - \tilde{U}_h \|_{0, R_K}$

$$
\| \Phi - \tilde{U}_h \|_{0, R_K} = \sum_{i=1}^m \| \Phi - \tilde{U}_h \|_{0, s_i} ,
$$

$$
\Phi(\tau) - \tilde{U}_h(\tau) = \int_0^{Q_1(\tau)} \frac{d}{dx_1} \left[ \Phi_i(x_1, 0) - \tilde{U}_h(x_1, 0) \right] \, dx_1 +
$$

$$
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+ \int_0^{P_2(x)} \frac{d}{dx_2} \left[ \Phi_i(Q_1(x), x_2) - \hat{U}_h(Q_1(\tau), x_2) \right] dx_2 = \\
= \int_0^{Q_1(t)} \frac{d}{dx_1} \left[ \Phi_i(x_1, 0) - \hat{U}_h(x_1, 0) \right] dx_1.

Since \( \psi \in C^3(\langle a, b \rangle) \), we have \( \hat{U}_h \in H^2(A_iA_{i+1}) \). Hence

\begin{equation}
|\Phi(\tau) - \hat{U}_h(\tau)|^2 \leq ch^3|\Phi_i - \hat{U}_h|_{L^2(\Omega \cap \Omega')}^2 \leq ch^3|\hat{U}_h|_{L^2(\Omega \cap \Omega')}^2. \tag{3.35}
\end{equation}

As \( \hat{U}_h(x) = (\hat{u}_h - \hat{u}_h'')(x) \cdot n(P(x)) \) and \( \hat{u}_h, \hat{u}_h'' \in P_1(A_iA_{i+1}) \), we may write:

\(|\hat{U}_h|_{L^2(A_iA_{i+1})}^2 \leq ch^3\left(\|u_h''\|_{L^2(A_iA_{i+1})}^2 + \|u_h''\|_{L^2(A_iA_{i+1})}^2\right)\).

Thus, (3.35), Lemma 3.3 on \( A_iA_{i+1} \) and the definition of strong regularity of \( \{\mathcal{T}_h\} \) yield:

\begin{equation}
\int_0^{T(\tau)} |\Phi(\tau) - \hat{U}_h(\tau)|^2 d\tau \leq ch^3\left(\|u_h''\|_{L^2(A_iA_{i+1})}^2 + \|u_h''\|_{L^2(A_iA_{i+1})}^2\right) \leq ch^3\left(\|u_h''\|_{L^2(A_iA_{i+1})}^2 + \|u_h''\|_{L^2(A_iA_{i+1})}^2\right). \tag{3.36}
\end{equation}

Adding (3.36) for \( i = 1, \ldots, m \) we obtain

\begin{equation}
\|\Phi - \hat{U}_h\|_{L^2(\Gamma)}^2 \leq ch^3\left(\|u_h''\|_{L^2(\Gamma)}^2 + \|u_h''\|_{L^2(\Gamma)}^2\right), \tag{3.37}
\end{equation}

where \( \Gamma = \bigcup_{i=1}^m A_iA_{i+1} \) is the polygonal approximation of \( \Gamma \). Using the trace theorem and Lemma 3.4, we obtain

\begin{align*}
\|u_h''\|_{L^2(\Gamma)}^2 &\leq c \|u_h''\|_{L^2(\Omega')}, \\
\|u_h''\|_{L^2(\Gamma)}^2 &\leq c \|u_h''\|_{L^2(\Omega')},
\end{align*}

using these estimates, (3.33), (3.34) and (3.37) we deduce

\begin{equation}
\|\Phi - \hat{U}_h\|_0^2 \leq ch^3/2\|u_h\|_{L^2(\Omega')}. \tag{3.38}
\end{equation}

Next let \( v \in V \) be such that \( v'' = 0 \) on \( \Omega' \) and \( v' \) such that \( v'.n = \Phi \) on \( \Gamma \). Then \( v'.n = v'' \cdot n = v'.n = \Phi \leq 0 \) on \( \Gamma \), consequently \( v \in \mathcal{X} \). Finally we may write

\begin{equation}
\int_{\Gamma} T_u(u) \left[ (v'' - u_h') - (v'' - u_h') \right] ds = \tag{3.39}
\end{equation}

1) The constant \( c \) such that \( \|u_h\|_{L^2(\Gamma)} \leq c \|u_h''\|_{L^2(\Omega')} \) and \( \|u_h''\|_{L^2(\Gamma)} \leq c \|u_h''\|_{L^2(\Omega')} \) respectively, depends on \( h \), in general. From the definition of the norm of traces however, (see [7], p. 88) it follows that \( c \) can be estimated independently on \( h \) for \( h > 0 \) sufficiently small.

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The assertion (3.30) follows from (3.31) and (3.39).

Remark 3.4. If (1.6) and (1.8) hold, the norms \( \| u_h \|_{1, \Omega} \) are bounded (see the proof of Theorem 1.2). The condition (1.8) follows from the coerciveness of \( \mathcal{L} \) over \( \mathcal{H} \), provided \( \mathcal{H}_h \subset \mathcal{H} \) for all \( h \). Unfortunately, this is not our case and therefore we have to assume the boundedness of norms in Theorem 3.3 explicitly.

Remark 3.5. The assumption on the boundedness of the norms \( \| u_h \|_{1, \Omega} \) is satisfied if e.g. on \( \Gamma_{k_0} \subset \Gamma_K \) (chosen a priori) the bilateral contact of \( \Omega' \) and \( \Omega'' \) is considered (see Remark 3.3). Then \( \mathcal{L} \) is coercive on \( V \), hence (1.8) holds. The condition (1.6) follows from Lemma 3.1, which can be proved also for curved domains, modifying slightly the proof.

Remark 3.6. In [1] (Theorem 3.4) some sufficient conditions for the coerciveness of \( \mathcal{L} \) on \( \mathcal{H} \) have been presented. Let \( \Gamma_K \) contain a straight-line segment \( I \). Let us define

\[
\mathcal{H}_I = \{ v \in V \mid v'_n - v''_n \leq 0 \text{ on } I \}
\]

Assume that the sufficient conditions, mentioned above, are also satisfied, if \( \mathcal{H} \) is replaced by \( \mathcal{H}_I \). (For instance, in the situation of Fig. 5 – [1], it holds \( \mathcal{H} \cap \mathcal{R} = \mathcal{H}_I \cap \mathcal{R} \) and the sufficient conditions become identical). Then \( \mathcal{L} \) is coercive on \( \mathcal{H}_I \), as follows from the proof of Th. 2.4 [1], substituting only \( \mathcal{H} \) by \( \mathcal{H}_I \) and \( \Gamma_K \) by \( I \) everywhere. Since

\[
\mathcal{H}_h \subset \mathcal{H}_I \quad \forall h \in (0, 1)
\]

(see the proof of Lemma 2.1), we have

\[
\| v_h \|_{1, \Omega} \to \infty, \quad v_h \in \mathcal{H}_h \Rightarrow \mathcal{L}(v_h) \to +\infty,
\]

i.e. (1.8). From this and (1.6) the boundedness of norms \( \| u_h \|_{1, \Omega} \) follows.

Remark 3.7. Let us consider the case, solved by Theorem 2.3 in [1]. Let us restrict ourselves to the problem \( (P_1) \), with \( \Gamma_0 \) and \( \Gamma_K \) parallel to \( x_1 \)-axis. Then (see [1], Fig. 4)

\[
\mathcal{R}_V = V \cap \mathcal{R} = \mathcal{H} \cap \mathcal{R} = \{ z = (z', z'') \mid z' = (0, 0), z'' = (a, 0), a \in \mathbb{R}^1 \},
\]

where \( \mathcal{R} \) is the subspace of rigid bodies displacements. Let \( V = H \oplus \mathcal{R}_V \) be the orthogonal decomposition of \( V \) with respect to the following scalar product:

\[
(u, v)_V = [u, v] + p(u) \cdot p(v),
\]

where \([u, v] = \int_{\Omega} \varepsilon_{ij}(u) \varepsilon_{ij}(v) \, dx\),

\[
p(v) = \int_{\Gamma''} v''_n \, ds, \quad \Gamma'' \subset \partial \Omega'', \quad \text{mes } \Gamma'' > 0.
\]
Then (see [1]) we have

\[ H = \{ v \in V \mid p(v) = 0 \} . \]

From Th. 2.3 in [1] it follows that it is sufficient to solve the problem \((\mathcal{P}_1)\) only on the set \(\hat{K} = \mathcal{X} \cap H\). Then the problem

\[ \begin{aligned} \text{find } & \hat{u} \in \hat{K} \text{ such that } \\
& L'(\hat{u}) \leq L'(v) \quad \forall v \in \hat{K} \end{aligned} \]

has a unique solution. At the same time, the closed convex sets \(\hat{K}_h = \mathcal{X}_h \cap H\) can easily be realized numerically. In fact,

\[ \hat{K}_h = \{ v \in \mathcal{X}_h \mid \int_{\Gamma} v'_i \, ds = 0 \} \subset \hat{K}, \]

because \(\Gamma_K\) is a straight-line segment. It means that only one supplementary condition

\[ \int_{\Gamma} v'_i \, ds = 0 \]

must be added to the definition of \(\mathcal{X}_h\). The problem

\[ \begin{aligned} \text{find } & \hat{u}_h \in \hat{K}_h \text{ such that } \\
& L'(\hat{u}_h) \leq L'(v) \quad \forall v \in \hat{K}_h \end{aligned} \]

has a unique solution \(\hat{u}_h\). The element \(u_h \in \mathcal{X}_h\) represents a solution of \((\mathcal{P}_1)\) if and only if

\[ u_h = \hat{u}_h + y, \quad y \in \mathcal{R}_V. \]

As the inequality of the Korn's type holds on \(H\), i.e.

\[ |v| \geq c \|v\|_{1,\Omega} \quad \forall v \in H, \]

the seminorm \(|\cdot|\) in the error estimate (3.13) can be replaced by the norm \(\|\|_{1,\Omega}\), if \((\mathcal{P}_1)\) and \((\mathcal{P}_{1h})\) are solved on \(\hat{K}\) and \(\hat{K}_h\), respectively (i.e. if we solve the problems \((\hat{P}_1)\) and \((\hat{P}_{1h})\)). However, the proof of rate of convergence requires a slight modification: \(v_h = u_t\) in the proof of Th. 3.1 must be replaced by \(v_h = P_H \hat{u}_t\), where \(\hat{u}_t\) is the linear Lagrange interpolate of \(u\) and \(P_H\) is the projection of \(V\) onto \(H\) (see [6], Th. 2.1).

### 3.2. CONTACT PROBLEMS WITH AN ENLARGING CONTACT ZONE

Let the weak solution \(u\) of the problem \((\mathcal{P}_2)\) be such that \(\varepsilon^M(u) \in \bar{Y}(\Omega^M), M = \gamma, \gamma^*\).

Then using the definition of \((\mathcal{P}_2)\) and (3.4) we obtain (cf. [1] — Th. 1.2) the same group of conditions (3.5)—(3.8) as in Section 3.1 and

\[ \begin{aligned} (3.9') & \quad u'_i - u'_i \leq \varepsilon \\
(3.10') & \quad - T_i^z(\cos \alpha)^{-1} = T_i^w(\cos \alpha)^{-1} \leq 0 \end{aligned} \]
Theorem 3.4. Let \((\mathcal{P}_2)\) and \((\mathcal{P}_{2h})\) have solutions \(u\) and \(u_h\), respectively, let \(u \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_e\), \(\tau^M(u) \in \tilde{Y}(M')\), \(M = ^\prime, ^\prime\prime\), \(u^i_\xi \in W^{1,p}(\Gamma_K), u^i_\xi \in W^{1,p}(\Gamma_K')\), and \(f', f'' \in C^2(\langle a, b \rangle)\). Moreover, let us suppose that the number of points on \(\Gamma_K, \Gamma_K'\), where the contact changes from binding to nonbinding, is finite. Then

\[
\begin{align*}
\left| u - u_h \right| & \leq c(u) h,
\end{align*}
\]

if the system \((\mathcal{F}_h)\), \(h \to 0^+\) is regular.

Proof. Analogously as in the proof of Theorem 3.3, using the Green's formula, we obtain

\[
\frac{1}{2} A(u - u_h, u - u_h) \leq 1/2 A(v_h - u, v_h - u) +
\]

\[
+ \int_{\Gamma_K} T'_\xi(u) (v'_{\xi} - u'_{\xi}) \, ds + \int_{\Gamma_K} T''_\xi(u) (v''_{\xi} - u''_{\xi}) \, ds +
\]

\[
+ \int_{\Gamma'_K} T'_\xi(u) (v'_{\xi} - u'_{\xi}) \, ds + \int_{\Gamma'_K} T''_\xi(u) (v''_{\xi} - u''_{\xi}) \, ds
\]

\[
\forall v_h \in \mathcal{H}_{eh}, \ \forall v \in \mathcal{H}_e.
\]

Let \(v_h = u_I\) be the \(P\)-interpolate of \(u\), constructed by means of the isoparametric technique, i.e.

\[
u_I|_{T} = \tilde{N}(u|_{T} \circ F_T) \circ F_T^{-1},
\]

where \(\tilde{N}\) denotes the operator of linear Lagrange interpolation on \(\tilde{T}\). It is easy to see that \(u_I \in \mathcal{H}_{eh}\). Using the approximative properties of \(u_I\) (cf. [3]) we can estimate \(A(u_I - u, u_I - u)\) as follows:

(3.40)

\[
|A(u_I - u, u_I - u)| \leq M\|u_I - u\|_{2, \Omega} \leq ch^2\|u\|_{2, \Omega}.
\]

We may write

\[
\begin{align*}
\int_{\Gamma_K} T'_\xi(u) (u'_{\xi} - u'_\xi) \, ds + \int_{\Gamma_K} T''_\xi(u) (u''_{\xi} - u''_{\xi}) \, ds &=
\int_a^b T'_\xi(u) \left[ (u''_{\xi} - u'_{\xi}) - (u''_{\xi} - u''_{\xi}) \right] \, d\eta,
\end{align*}
\]

where \(T'_\xi(u) \equiv T'_\xi(u) (\cos \alpha_I)^{-1} = -T'_\xi(u) (\cos \alpha_I)^{-1}\). Set

\[
W_\eta(\eta) = u'_{\xi}(f''(\eta), \eta) - u'_{\xi}(f'(\eta), \eta)
\]

\[
\Psi(\eta) = u''_\xi(f''(\eta), \eta) - u''_\xi(f'(\eta), \eta) \ \eta \in \langle a, b \rangle.
\]
From the definition of \( u'_j(f'(\eta), \eta), u''_j(f''(\eta), \eta) \) it follows that these are piecewise linear functions of \( \eta \)-variable on \( \langle a, b \rangle \) with the nodes of \( C_j \) (see the construction of \( \mathcal{X}_h \), where suitable multiples of \( \eta \) are chosen as the parameters in the arc representation). Since \( \xi \) is a fixed direction, \( W_h \) is also a piecewise linear function on \( \langle a, b \rangle \).

Let
\[
\Gamma^0 = \{ \eta \in \langle a, b \rangle \mid u'_\xi - u'_\eta = \epsilon \},
\]
\[
\Gamma^- = \{ \eta \in \langle a, b \rangle \mid u''_\xi - u''_\eta < \epsilon \}.
\]

If \( \langle C_i, C_{i+1} \rangle \subseteq \Gamma^0 \) (see Fig. 2) then \( W_h \) is the linear Lagrange interpolate of \( \varepsilon \) on \( \langle C_i, C_{i+1} \rangle \) and

\[
(3.41) \int_{C_i}^{C_{i+1}} T_\xi(u) (W_h(\eta) - \mathcal{U}(\eta)) \, d\eta = \int_{C_i}^{C_{i+1}} T_\xi(u) (W_h - \varepsilon) \, d\eta \leq ch^2 |_{\xi,C_{i+1},C_i} \, ,
\]

If \( \langle C_i, C_{i+1} \rangle \subseteq \Gamma^- \), then \( T_\xi(u) \equiv 0 \) on \( \langle C_i, C_{i+1} \rangle \). Hence

\[
(3.42) \int_{C_i}^{C_{i+1}} T_\xi(u) (W_h(\eta) - \mathcal{U}(\eta)) \, d\eta = 0 .
\]

Let \( \mathcal{J} \) be the system of all \( \langle C_i, C_{i+1} \rangle \subseteq \langle a, b \rangle \), containing both points of \( \Gamma^0 \) and \( \Gamma^- \). Using the assumption of the Theorem, we have

\[
(3.43) \int_{C_i}^{C_{i+1}} T_\xi(u) (W_h(\eta) - \mathcal{U}(\eta)) \, d\eta \leq h \| T_\xi(u) \|_{\infty, \langle C_i, C_{i+1} \rangle} \| W_h - \mathcal{U} \|_{\infty, \langle C_i, C_{i+1} \rangle} \leq ch^2 \| T_\xi(u) \|_{\infty, \langle C_i, C_{i+1} \rangle} |\mathcal{U}|_{1, \infty, \langle C_i, C_{i+1} \rangle} .
\]

Due to the assumptions, the number of all \( \langle C_i, C_{i+1} \rangle \in \mathcal{J} \) can be bounded from above independently of \( h \). From (3.41) — (3.43) we obtain

\[
(3.44) \int_a^b T_\xi(u) [(u''_\xi - u''_\eta) - (u''_\xi - u''_\eta)] \, d\eta \leq c(u) h^2 .
\]

It remains to estimate
\[
\int_{\mathcal{X}_h} T_\xi(u) (v'_\xi - u'_\eta) \, ds + \int_{\mathcal{X}_h} T''_\xi(u) (v''_\xi - u''_\eta) \, ds = \int_a^b T_\xi(u) [(v''_\xi - v''_\xi) - (u''_\xi - u''_\eta)] \, d\eta , \quad v \in \mathcal{X}_h .
\]

Let us denote
\[
\mathcal{U}_h(\eta) = u''_\xi(\eta, f'(\eta)) - u''_\eta(\eta, f'(\eta)) , \quad \eta \in \langle a, b \rangle
\]
and define the function \( W_h \) as follows
\[
W_h(\eta) = \inf_{\eta \in \langle a, b \rangle} [\mathcal{U}_h(\eta), \varepsilon(\eta)] .
\]

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It is readily seen that $W_h \in H^1(\langle a, b \rangle)$ and $W_h \leq \varepsilon$ on $\langle a, b \rangle$. Since

$$W_h - \mathcal{U}_h = \begin{cases} 0 & \text{if } \mathcal{U}_h \leq \varepsilon \\ \varepsilon - \mathcal{U}_h & \text{if } \mathcal{U}_h > \varepsilon \end{cases}$$

we can write

$$\int_a^b T_\varepsilon(u) (W_h - W_h) \, d\eta \leq c \|W_h - \varepsilon\|_{0, \delta},$$

where $\delta \in \langle a, b \rangle$ is the set of points, where $\mathcal{U}_h > \varepsilon$. As $\mathcal{U}_h(C_j) \leq e_j$, $j = 1, \ldots, m$ by the definition of $\mathcal{K}_\varepsilon$, we have also $\mathcal{U}_h(C_j) \leq \varepsilon_j$, where $\varepsilon_j$ is the piecewise linear interpolate of $\varepsilon$ on $\langle a, b \rangle$. $\mathcal{U}_h$ and $\varepsilon_j$ are piecewise linear on $\langle a, b \rangle$, therefore $\mathcal{U}_h \leq \varepsilon_j$ on $\langle a, b \rangle$. Hence (3.45) can be written in the following form:

$$\int_a^b T_\varepsilon(u) (W_h - W_h) \, d\eta \leq C \|\varepsilon - \varepsilon\|_{0, \delta} \leq C \|\varepsilon - \varepsilon\|_{0, \langle a, b \rangle} \leq ch^2\|\varepsilon\|_{2, \langle a, b \rangle}.$$

The rest of the proof can be accomplished in the same manner as that for Theorem 3.3. There exists a $v \in V$ such that $v^\prime = 0$ on $\Omega^\prime$ and $-v^\prime = W_h$. Then $v \in \mathcal{K}_\varepsilon$ and it holds:

$$\int_a^b T_\varepsilon(u) [v^\varepsilon - u^\varepsilon - (u^\varepsilon_h - u^\varepsilon)] \, d\eta =$$

$$= \int_a^b T_\varepsilon(u) [W_h - \mathcal{U}_h] \, d\eta \leq ch^2\|\varepsilon\|_{2, \langle a, b \rangle}.$$

Using also (3.40), (3.44), the assertion of the Theorem now follows.

As in the case of contact problems with a bounded contact zone, we shall prove the convergence of approximate solutions $u_h$ to the solution $u$ of the problem $(\Phi_2)$ without any regularity assumptions. To this end, we need some auxiliary lemmas.

**Lemma 3.5.** Suppose that $f^M \in C^M, M = \prime, \prime', m \geq 3, \Gamma_k^M \cap \Gamma_u = 0, \Gamma_k^M \cap \Gamma_0 = 0$ and there exists only a finite number of points $\Gamma_u \cap \Gamma_0, \Gamma_0 \cap \Gamma_0$. Let $u \in \mathcal{K}_\varepsilon$ satisfy the condition $u^\varepsilon_h - u^\varepsilon_h \leq f^\prime - f^\prime$ in $\langle a - \delta, b + \delta \rangle$, with some $\delta > 0$.

Then $u$ belongs to the closure (in $W$) of the set

$$\mathcal{K}_\varepsilon \cap [C^m(\Omega^\prime)]^2 \times [C^m(\Omega^\prime)]^2.$$

**Proof.** Consider a system of open domains $\{B_{ij}\}_{i=0}^k$ covering $\Omega^\prime \cup \Omega^\prime$ and such that $\bar{B}_0 \subset \Omega^\prime, \bar{B}_1 \subset \Omega^\prime, \Gamma_k^\prime \cup \Gamma_k^\prime \subset \bigcup_{j=2}^k B_j, (\Gamma_k^\prime \cup \Gamma_k^\prime) \cap B_i \neq \emptyset \Leftrightarrow 2 \leq i \leq k$. Assume that the union of arcs (see Fig. 6)

$$\hat{PQ}^\prime \cup \hat{PQ}^\prime, \quad Q^M = (f^M(b), b), \quad M = \prime, \prime'$$

is contained in one and only one domain $B_j$; the other domains contain at most one singular point (vertex or point of $\Gamma_u \cap \Gamma_0, \Gamma_0 \cap \Gamma_0$).
We use the decomposition of unity as in the proof of Lemma 3.1. and construct smooth approximations of each \( u_j = u \phi_j \). In general, we can proceed like in the proof of Lemma 3.1, except for the situation of Fig. 6., where we argue as follows.

\[ \dot{x} = T^M x : \{ \dot{\xi}^M = \xi - f^M(\eta), \ \dot{\eta}^M = \eta \}, \ M = \cdot, , \cdot. \]

\[ \dot{x} = (\dot{\xi}, \dot{\eta}), \ x = (\xi, \eta). \]

Denote \( \dot{B} = T'(\Omega' \cap B) \cup T''(\Omega'' \cap B) \) and \( \dot{u}^M(\dot{x}) = u^M((T^M)^{-1} \dot{x}) \). Since

\[ u^M_f(\eta, \xi) - u^M_0(\eta, \xi) - e(\eta) \leq 0, \eta_0 < \eta \leq b, \]

we have

\[ u(\eta) \equiv (\dot{u}^M - \dot{u}_0 - \dot{b}) \leq 0 \] for \( \xi + 0, \ \eta_0 < \eta \leq b \).

Let us extend \( \dot{b} \) onto the interval \( b < \dot{\eta} \) in such a way that the extension \( E\dot{b} \in C^m, \ E\dot{u} \) remains non-positive, \( E\dot{u} \subset H^{1/2} \) and \( \text{supp} \ E\dot{u} \subset \dot{B}. \)

Then there exists a function \( \dot{b} \in H^1(\dot{B}) \) such that \( \dot{b} \leq 0 \) in \( \dot{B}, \dot{b} = E\dot{u} \) for \( \dot{\xi} = 0, \text{supp} \dot{b} \subset \dot{B}. \)

If we extend \( \dot{u}_\xi^M \) across the \( \dot{\eta} \)-axis to get functions \( E\dot{u}_\xi^M \) even in \( \dot{\xi} \), we may write

\[ E\dot{u}_\xi^M - E\dot{u}_\xi^0 - E\dot{b} = \dot{\theta} + \dot{\xi}, \]

\[ \dot{\xi} \in H^1(\dot{B}), \ \dot{\xi}|_{\xi = 0} = 0. \]

Regularizing \( \dot{b} \) and \( \dot{\xi} \), we obtain

\[ (R_\xi \dot{b} + \dot{\xi})|_{\xi = 0} \leq 0, \ R_\xi \dot{b} + \dot{\xi} \rightarrow \dot{\theta} + \dot{\xi} \ in \ H^1(\dot{B}). \]

Define

\[ \dot{u}_x^\xi = R_\xi E\dot{u}_\xi^M|_{T \Omega'} \]

\[ \dot{u}_\xi^x = [R_\xi E\dot{u}_\xi^M - R_\xi \dot{b} - \dot{\xi}|_{T \Omega'} - \dot{\theta}, \]

\[ u_\xi^x = \dot{u}_x^\xi(\xi - f'(\eta), \eta), \ u_\xi^x = \dot{u}_x^\xi(\xi - f'(\eta), \eta), \ (\dot{\xi} = e). \]

\[ 1 \) We can take \( E\dot{b} = \dot{\phi} f'(f' - f''), \) provided that the point \( (0, b + \dot{\xi}) \) is outside \( \dot{B}. \)
Then the condition (3.46) is satisfied by $u_{\xi_0}, u^M_\xi \in C^m$ and $\|u_{\xi_0}^M - u_\xi^M\|_{1, Q^M} \to 0$, since both $T^M$ and $(T^M)^{-1}$ are Lipschitz mappings.

Lemma 3.6. Let $\varphi$ be a continuous function defined on $\langle a, b \rangle$, $(-\infty < a < b < \infty)$, $D_n: a = x_0^n < x_1^n < \ldots < x_n^n = b$ a division of $\langle a, b \rangle$, $v(D_n) = \max_{i=1, \ldots, n} |x_i^n - x_{i-1}^n| \to 0$ for $n \to \infty$. Let $\{\psi_n\}_{n=1}^\infty$ be a sequence of piecewise linear functions with nodes at $x_i^n$ such that $\psi_n(x_i^n) \leq \varphi(x_i^n) \forall i = 0, \ldots, n$; $n = 1, 2, \ldots$. Let $\psi_n \to \psi$ a.e. in $\langle a, b \rangle$. Then $\psi \leq \varphi$ a.e. in $\langle a, b \rangle$.


Theorem 3.5. Let the problem $(\mathcal{P}_2)$ have precisely one solution $u$ and the norms $\|u_h\|$ of solutions of the problems $(\mathcal{P}_{2h})$ remain bounded. Let all assumptions of Lemma 3.5 be satisfied. Then for any regular system of triangulations $\mathcal{T}_h$ we have

$$\|u - u_h\|_{1, \Omega} \to 0, \quad h \to 0^+.$$ 

Proof. We must verify (1.6'), (1.7) and use Remark 1.6. From Lemma 3.5, using the same arguments as in the proof of Theorem 3.2, we obtain (1.6'). It remains to verify (1.7). Let $v_h \in \mathcal{K}_{eh}$ be such that $v_h \to v$ in $\mathcal{K}^1(\Omega)$. By virtue of the complete continuity of the trace mapping we obtain

$$v_h' \to v' \text{ in } L^2(\Gamma_h^1), \quad v_h'' \to v'' \text{ in } L^2(\Gamma_h^1) \text{ (strongly)}.$$

Hence subsequences of $v_h'$ and $v_h''$ exist such that

$$V_h(\eta) = v_h''(f''(\eta), \eta) - v_h'(f'(\eta), \eta) \to v''(f''(\eta), \eta) - v'(f'(\eta), \eta) \equiv V(\eta)$$

a.e. in $\langle a, b \rangle$. Since $V_h(\eta)$ is piecewise linear on $\langle a, b \rangle$ and $V_h(C_i) \leq \varepsilon(C_i)$, $i = 1, 2, \ldots$, Lemma 3.6 implies $V \leq \varepsilon$ a.e. in $\langle a, b \rangle$.

References


**Souhrn**

**KONTAKTNÍ PROBLÉM PRUŽNÝCH TĚLES. ČÁST II.: APROXIMACE METODOU KONEČNÝCH PRVKŮ**

**Jaroslav Haslinger, Ivan Hlaváček**

Práce se zabývá aproximací kontaktního problému dvou rovinných pružných těles metodou konečných prvků. Navazuje bezprostředně na předchozí výsledky autorů, obsažené v [1]. Přípustná konvexní množina posunutí pro klasický variační princip se aproximuje po částech lineárními vektorovými funkcemi na trojúhelnících. Studuje se rychlost konvergence a konvergence přibližných řešení k řešení přesnému v závislosti na normě dělení. Uvažují se přitom jednak úlohy, v nichž se rozsah kontaktu během deformace nemění, jednak úlohy s proměnným rozsahem kontaktu.

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