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Aplikace matematiky, Vol. 26 (1981), No. 6, 437--448

Persistent URL: http://dml.cz/dmlcz/103934

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ON THE SOLUTION OF A GENERALIZED SYSTEM
OF VON KÁRMÁN EQUATIONS

JOZEF Kačur

(Received January 21, 1980)

INTRODUCTION

A nonlinear system of equations generalizing von Kármán equations is studied. The system considered is derived in [1] under the assumption of a nonlinear relation between the intensity of stresses and deformations in the constitutive law \(\sigma_i = E(1 - \omega)\) and stands as a model for large deformations of thin plates or shells. In the case \(\omega = 0\) this system reduces to the system von of Kármán equations. The function \(\omega = \omega(e)\) can also characterize the plasticity properties of the given material but the derived system is a model for large deformations of elastic-plastic plates for simple exterior stresses only (i.e. all exterior stresses arise from zero stresses in a monotonic way). From the numerical point of view the generalized system has been analysed also in [2]. The case \(\omega = \omega(x, y)\) has been considered in [8]. Our goal is to prove the existence of a solution and its properties for \(\omega \to 0\). We use the technique developed in [3–6] and some results from [7].

1. NOTATION AND FORMULATION OF THE PROBLEM

Let \(\Omega \subset R^2\) be a simply connected bounded domain describing the shape of a plate. We assume that the boundary \(\partial \Omega\) is piecewise three times continuously differentiable (see [5]). Denote \(w_x = \partial w/\partial x\), \(w_y = \partial w/\partial y\), \(w_{xy} = (w_x)_y\) etc.; \(A^2w = w_{xxxx} + 2w_{xxyy} + w_{yyyy}\); \([w, f] = w_{xx}f_{yy} + w_{yy}f_{xx} - 2w_{xy}f_{xy}\); \(w_v\) stands for the outward normal derivative with respect to \(\partial \Omega\). By means of (from the constitutive law) we define the functions \(a_i\) \((i = 1, 2, 3)\) in the following way:

\[
Q_1 = \frac{2}{h} \int_{-h/2}^{h/2} \omega \, dz, \quad Q_2 = \frac{4}{h^2} \int_{-h/2}^{h/2} z \omega \, dz, \quad Q_3 = \frac{8}{h^3} \int_{-h/2}^{h/2} z^2 \omega \, dz
\]

\[
a_1 = (1 - \frac{1}{2}Q_1)^{-1}, \quad a_2 = a_1 Q_2, \quad a_3 = \frac{1}{2}(2Q_3 + a_1 Q_2^2),
\]

where \(h\) is the thickness of the plate. Let \(w\) be the deflection and \(F\) Airy’s stress function of the plate. Then \(a_i\) are the functions of \(w_{xx}, w_{xy}, w_{yy}, F_{xx}, F_{xy}\) and \(F_{yy}\).
We assume \(a_t\) to be in the form \(a_t = a_t(x, y, w, w_x, w_y, w_{xx}, w_{xy}, w_{yy}; F, F_x, F_y, F_{xx}, F_{xy}, F_{yy})\). A corresponding system for unknown functions \(F, w\), derived in [1] under the nonlinear constitutive law, is of the form

\[
\begin{align*}
\Delta^2 w - \left( (F_{xx} + \frac{1}{2} F_{yy}) a_3(Dw; DF) \right)_{xx} - \left( (F_{yy} + \frac{1}{2} F_{xx}) a_3(Dv; DF) \right)_{yy} - \\
- (w_{xy} a_3(Dw; DF))_{xy} + \frac{9}{4Eh} \left\{ \left( (F_{yy} a_2(Dw; DF))_{xx} + ((F_{xx} a_2(Dw; DF))_{yy} - \\
- 2(F_{xy} a_2(Dw; DF))_{xy} \right) \right\} = \frac{9}{Eh^2} \left[ F, w \right] + \frac{q}{P},
\end{align*}
\]

\[
\begin{align*}
((F_{xx} - \frac{1}{2} F_{yy}) a_1(Dw; DF))_{xx} + ((F_{yy} - \frac{1}{2} F_{xx}) a_1(Dw; DF))_{yy} + \\
+ 3(F_{xy} a_1(Dw; DF))_{xy} - \frac{Eh}{4} \left\{ (w_{xx} a_2(Dw; DF))_{yy} + (w_{yy} a_2(Dw; DF))_{xx} - \\
- 2(w_{xy} a_2(Dw; DF))_{xy} \right\} = - \frac{E}{2} \left[ w, w \right]
\end{align*}
\]

for \((x, y) \in \Omega\), where \(E\) is the modulus of elasticity, \(P = \frac{1}{2} Eh^3\) and \(q\) is the density of the perpendicular load.

Together with \((E_1), (E_2)\) we consider the following boundary conditions

\[
\begin{align*}
(B) \quad w = w_v = 0 \quad \text{on} \quad \partial \Omega \quad \text{and} \quad F = F_0, \quad F_v = F_{0,v} \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \(F_0 \in C^2(\overline{\Omega})\) is a given function.

Let \(\zeta : \overline{\Omega} \to [0, 1]\) be an arbitrary function with the property

\[
(P) \quad \zeta \in C^2(\overline{\Omega}) \quad \text{and} \quad \zeta = 1, \quad \zeta_v = 0 \quad \text{on} \quad \partial \Omega.
\]

We denote \(f_0 = \zeta F_0\) and we consider \(F\) in the form \(F = f + f_0\), where \(f = f_v = 0\) on \(\partial \Omega\).

For the sake of simplicity we denote \((u, v) = \int_\Omega (u v_x v_{xx} + 2 u v_x v_{xy} + u v_y v_{yy}) \). dx dy, \((u, v) = \int_\Omega u v \) dx dy and \(B(u; v, w) = \int_\Omega (u v_x v_{xy} + u v_y v_{xx} - u_x v_y v_{xx} - \)

\( w_{xx} v_y v_y) \) dx dy for \( u, v, w \in W^2_2(\Omega) \) (Sobolev space).

**Definition.** A couple \(\{w, F\}\) is said to be a variational solution of \((E_1), (E_2), (B)\), iff \(w, F - f_0 \in W^2_2(\Omega)\) and the identities

\[
\begin{align*}
((L_1(w, F), \varphi)) = (w, \varphi)_w - ((w_{xx} + \frac{1}{2} w_{yy}) a_3(Dw; DF), \varphi_{xx}) - \\
- ((w_{xy} + \frac{1}{2} w_{xx}) a_3(Dw; DF), \varphi_{xy}) - (w_{rx} a_3(Dw; DF), \varphi_{xx}) + \\
+ \frac{9}{4Eh} \left\{ ((F_{xy} a_2(Dw; DF), \varphi_{xx}) + (F_{xx} a_2(Dw; DF), \varphi_{yy}) - \\
- 2(F_{xy} a_2(Dw; DF), \varphi_{xy}) \right\} - \frac{9}{Eh^2} B(w; F, \varphi) = \left( \frac{q}{P}, \varphi \right),
\end{align*}
\]
\[(L_2(w, F), \psi) \equiv ((F_{xx} - \frac{1}{2} F_{yy}) a_1(Dw; DF), \psi_{xx}) + ((F_{yy} - \frac{1}{2} F_{xx}) a_1(Dw; DF), \psi_{yy}) + 3(F_{xy} a_1(Dw; DF), \psi_{xy}) - \frac{1}{4} E h \{(w_{xx} a_2(Dw; DF), \psi_{yy}) + (w_{yy} a_2(Dw; DF), \psi_{xx}) - 2(w_{xy} a_2(Dw; DF), \psi_{xy}) + \frac{1}{2} E B(w; w, \psi) = 0 \]

hold for all \( \varphi, \psi \in \hat{W}_2^2(\Omega) \).

Using Green’s theorem in (1) and (2) we can easily find that a variational solution of \((E_1), (E_2), (B)\) is also a classical solution under the regularity assumptions on \( w, F \) and \( a_i \) (\( i = 1, 2, 3 \)).

The expression \( B(u; v, w) \) in (1) and (2) is well defined for \( u, v, w \in W_2^2(\Omega) \) since the inequality
\[(3) \quad |B(u; v, w)| \leq \|u\|_{W_2^2} \|v\|_{W_2^2} \|w\|_{W_2^2},\]
holds. Moreover, for \( u, v \in W_2^2(\Omega) \) and \( w \in \hat{W}_2^2(\Omega) \) we have
\[(4) \quad B(w; u, v) = B(v; u, w) = B(v; w, u) \]
(see, e.g., [3]).

2. EXISTENCE OF A SOLUTION

We prove the existence of a variational solution of the problem \((E_1), (E_2), (B)\) using the abstract existence results for the corresponding operator equation \( Au = G \). We deduce this equation in the following way: Let us denote \( H = \hat{W}_2^2(\Omega) \times \hat{W}_2^2(\Omega) \) with the usual norm \( \| \cdot \|_H \). Let \( u = \{ w, f \}, v = \{ \varphi, \psi \} \in H \). We define the operator
\( A_{\zeta}: H \to H^*(H^* \equiv W_2^{-2} \times W_2^{-2}) \) by means of the form \( \langle A_{\zeta} u, v \rangle = \langle L_1(w, f + f_0), \psi \rangle + \langle L_2(w, f + f_0), \psi \rangle \) since \( f_0 = \zeta_0 \) and \( \zeta \) is a function with the property \((P)\). In what follows we omit the index \( \zeta \) in \( A_{\zeta} \). \( G \in H^* \) is of the form \( \{ g/P, 0 \} \). Clearly, the solvability of \( Au = G \) in \( H \) is equivalent to the existence of a variational solution of \((E_1), (E_2), (B)\).

Under certain assumptions on \( a_i \) (\( i = 1, 2, 3 \)) we prove that \( A : H \to H^* \) is a continuous, bounded operator with the property \( S \) (i.e., \( u_n \rightharpoonup u \) (weak convergence) and \( \langle Au_n - Au, u_n - u \rangle \to 0 \) implies \( \| u_n - u \|_H \to 0 \)). Using the result from [5] (see [3], [6]), under a suitable choice of the function \( \zeta \) we prove coercivity of the operator \( A(A \equiv A_{\zeta}) \). Then from well known results (see, e.g., [7]) we obtain \( A(H) = H^* \), which implies the existence of a variational solution of \((E_1), (E_2), (B)\).

We assume that \( a_i(x, y, \xi, \tau) \) (\( i = 1, 2, 3 \)) are continuous functions in all variables defined on \( \Omega \times R^6 \times R^6 \), where the real vectors \( \xi, \tau \in R^6 \) stand instead of \( w, f \) and their derivatives up to the second order. We shall assume that there exist positive
constants $M_0$ and $M_i$ ($i = 1, 2, 3$) such that

$$a_i(x, y, \xi, \tau) \geq M_0,$$

$$|a_i(x, y, \xi, \tau)| \leq M_i, \quad i = 1, 2, 3,$$

for all $(x, y) \in \Omega$ and $\xi, \tau \in \mathbb{R}^6$.

Moreover, we shall assume that the partial derivatives $\partial a_i/\partial \xi_j$ and $\partial a_i/\partial \tau_j$ are continuous on $\Omega \times \mathbb{R}^6 \times \mathbb{R}^6$ for all $i = 1, 2, 3$ and $|j| \leq 2$ where $j$ is the multiindex $j = (j_1, j_2), j_1, j_2 \geq 0$ and $|j| = j_1 + j_2$. To prove the property $A$ of the operator $A$ we shall assume that there exist $C_j \geq 0$ ($|j| \leq 2$) and $s > 1$ such that the estimates

$$\left| \frac{\partial a_i(x, y, \xi, \tau)}{\partial \xi_j} \right| + \left| \frac{\partial a_i(x, y, \xi, \tau)}{\partial \tau_j} \right| \leq \frac{C_j}{1 + \sum_{|j| = 2} (|\xi_j|^s + |\tau_j|^s)}$$

hold for all $i = 1, 2, 3, |j| \leq 2$, $(x, y) \in \Omega$ and $\xi, \tau \in \mathbb{R}^6$.

**Lemma 1.** Let (6) be satisfied. Then the operator $A$ is continuous and bounded from $H$ into $H^\ast$.

**Proof.** Suppose $u_n \to u$ in $H$. It suffices to prove

$$\sup_{\|v\|_H \leq 1} \left| \langle Au_n, v \rangle - \langle Au, v \rangle \right| \to 0 \quad \text{for} \quad n \to \infty$$

and $\sup_{\|v\|_H \leq 1} \left| \langle Au, v \rangle \right| \leq C_D < \infty$ for $u$ from a bounded set $D$ in $H$. Denote $u_n = \{w_n, f_n\}$, $u = \{w, f\}$ and $v = \{\varphi, \psi\}$. We have $w_n \to w, f_n \to f$ in $W \equiv W^2_2(\Omega)$. Let us estimate the members of the type

$$I_n^{(1)} = \sup_{\|\varphi\|_W \leq 1} \left| B(w_n; f_n + f_0, \varphi) - B(w; f + f_0, \varphi) \right| \leq$$

$$\sup_{\|\varphi\|_W \leq 1} \left| B(w_n - w; f_n + f_0, \varphi) \right| + \sup_{\|\varphi\|_W \leq 1} \left| B(w; f_n - f, \varphi) \right| .$$

Owing to (3) we obtain $I_n^{(1)} \to 0$ for $n \to \infty$. Now we estimate the members of the type

$$I_n^{(2)} = \sup_{\|\varphi\|_W \leq 1} \left| ((w_n)_{x_1} a_3(Dw_n; D(f_n + f_0)) - w_{xx} a_3(Dw; D(f + f_0)), \varphi_{xx}) \right| .$$

From the relations

$$(w_n)_{x_1} a_3(Dw_n; D(f_n + f_0)) - w_{xx} a_3(Dw; D(f + f_0)) =$$

$$(w_n - w)_{x_1} a_3(Dw_n; D(f_n + f_0)) + w_{xx} a_3(Dw_n; D(f_n + f_0)) -$$

$$- w_{xx} a_3(Dw; D(f + f_0)) ,$$

and $u_n \to u$ in $H$ and (6) we easily deduce that $I_n^{(2)} \to 0$ for $n \to \infty$. From these facts we easily conclude (8). Boundedness of the operator can be proved analogously.
The coercivity of the operator \( A \) \((A = A^r)\) is proved by means of the result in [5] (see [3], [6]), which is based on the idea of Knightly [6], for a special choice of the function \( \zeta \).

**Lemma 2.** Suppose (5), (6). If the inequality

\[
\frac{3}{2} M_3 + 81 M_0^{-1} M_2^2 < 1
\]

is satisfied then there exists a \( \zeta \in C^2(\Omega) \) with the property \((P)\) and constants \( C_1, C_2 \) \((C_1 \equiv C_1(\zeta) > 0, \ C_2 \equiv C_2(\zeta) > 0)\) such that the estimate

\[
\langle Au, u \rangle \geq C_1 \|u\|^2_H - C_2
\]

holds for all \( u \in H \).

**Proof.** Let us put \( u = \{w, f\} \) into (1), (2). Using (4) and eliminating \( B(w; w, f) \) from (1), (2) we successively obtain the estimate

\[
\langle Au, u \rangle \geq \|w\|^2_w \left( 1 - \frac{3}{2} M_3 - \frac{9 M_3 L^2}{2 Eh} - \frac{9 M_2 e^2}{4 Eh} \right) + \frac{17}{E^2 h^2} \|f + f_0\|^2_w \left( \frac{M_0}{2} - \frac{9 Eh M_2}{2 L^2} - \frac{E^2 h^2 M_4 e^2}{12} \right) - \frac{9}{E h^2} B(w; f_0, w) - C(\varepsilon) \cdot \|f_0\|^2_{w_2^2},
\]

where \( L > 0 \) is an arbitrary number \( C(\varepsilon) \to \infty \) for \( \varepsilon \to 0 \), \( f_0 = \xi F_0 \) (see (B)) and \( \|v\|^2_w = \|v_x\|^2_{L_2} + \|v_y\|^2_{L_2} + 2 \|v_y\|^2_{L_2} \). In (11) Young’s inequality \( (ab \leq 2^{-1} e^{-2} a^2 + 2^{-1} e^{-2} b^2) \) has been used. Let us take \( L^2 = (M_0 - \gamma)^{1/2} 9 Eh M_2 \) where \( (0 < \gamma < M_0/2) \) is sufficiently small. Then owing to (9) we have

\[ C_0 = 1 - \frac{3}{2} M_3 - \frac{9 M_3 L^2}{2 Eh} > 0 \quad \text{and} \quad \frac{M_0}{2} - \frac{9 Eh M_2}{2 L^2} > 0. \]

Using the result from [5] (see also [3], [6]) we can choose such a \( \zeta \) with the property \((P)\) that the estimate

\[
|B(w; \xi F_0, w)| < \frac{C_0}{4} \|w\|^2_w
\]

holds. From (11), (12) and for sufficiently small \( \varepsilon \) we obtain the estimate (10) and Lemma 2 is proved.

Henceforth let \( \zeta \in C^2(\Omega) \) be a fixed function for which Lemma 2 holds true. In order to prove the property \( S \) for \( A \) we use the following lemma.

**Lemma 3.** Let \( a = (a_i), b = (b_i), A = (A_i), B = (B_i) \) be real vectors in \( E^n \). If \( s > 1 \) then there exists a constant \( K > 0 \) (independent of \( a, b, A, B \)) such that the
estimates
\[ I_i = \int_0^1 \frac{|a_i| + |b_i|}{1 + |a + t(A - a)|^s + |b + t(B - b)|^s} \, dt \leq K \]
hold for all \( i = 1, 2, \ldots, n \).

Proof. Denote \( x = a_i, y = A_i \). We assume \( x, y \geq 0 \).

For \( 0 \leq x \leq y \) we have
\[ I_i \leq I \equiv \int_0^1 \frac{x}{1 + (x + t(y - x))^s} \, dt = \frac{x}{y - x} \int_x^y \frac{dz}{1 + z^s} \leq \frac{x}{1 + x^s} \leq 1. \]

If \( x \leq 1 \) then \( I \leq 1 \). Thus we assume \( x \geq 1 \). For \( 0 \leq y \leq x \) we consider the cases 1) \( 0 \leq y \leq \frac{1}{2}x \) and 2) \( x \geq y \geq \frac{1}{2}x \). In the case 1) we have
\[ I \leq 2 \int_0^\infty \frac{dz}{1 + z^s} = 2K_s = \frac{2\pi}{s} \left( \sin \frac{\pi}{s} \right)^{-1}. \]

In the case 2) we have
\[ I \leq \frac{x}{1 + y^s} \leq \frac{x}{1 + 2^{-s}x^s} \leq 2. \]

Analogously, for \( y < 0, x \geq 0 \) we obtain \( I \leq 2K_s \). Hence Lemma 3 is proved with \( K = 4 \max (K_s, 1) \).

Denote
\[ C = K \left( 14 + \frac{21}{Eh} + 3Eh \right), \quad \delta = \max \{ C_i \}, \]
where \( K \) is from Lemma 3 and \( C_j \) are from (7). Our main lemma is

**Lemma 4.** Let (5)–(7) be satisfied. If the inequalities
\[ M_3 < \frac{3}{2}, \quad 1 - \frac{3}{2}M_3 + \frac{M_0}{2} - \left( \left( 1 - \frac{3}{2}M_3 - \frac{M_0}{2} \right)^2 + 4M_0^2 \left( \frac{9}{8Eh} + \frac{Eh}{8} \right) \right)^{1/2} > 2C \delta \]
hold then the operator \( A \) possesses the property \( S \).

Proof. Let \( u_n = \{ w_n, f_n \}, u = \{ w, f \} \in H \) and \( u_n \to u, P_n \equiv \langle Au_n - Au, u_n - u \rangle \to 0 \) for \( n \to \infty \). For simplicity we denote \( F_n = f_n + f_0, F = f + f_0 \). \( a_i(n) \equiv a_i(Dw_n; DF_n) \) and \( a_i(0) \equiv a_i(Dw; DF) \) (\( i = 1, 2, 3 \)). Using Young's inequality we
successively estimate

\[ P_n \geq \| w_n - w \|_W^2 - \frac{1}{2} M_3 \| w_n - w \|_W^2 - ((a_3(n) - a_3(0)) (w_{xx} + \frac{1}{2} w_{yy}),
(w_n - w)_{xx} - ((a_3(n) - a_3(0)) (w_{yy} + \frac{1}{2} w_{xx}), (w_n - w)_{yy} -
- ((a_3(n) - a_3(0)) w_{xy}, (w_n - w)_{xy}) - \frac{L_1^2 M_2^2}{8 Eh} \| w_n - w \|_W^2 - \frac{9 M_2}{8 Eh L_1^2} \| f_n - f \|_W^2 +\]

\[ + \frac{9}{4 Eh} \left\{ (F_{yy}(a_2(n) - a_2(0)), (w_n - w)_{xx}) + (F_{xx}(a_2(n) - a_2(0)), (w_n - w)_{yy}) -
- 2(F_{xy}(a_2(n) - a_2(0)), (w_n - w)_{xy}) \right\} + \frac{M_0}{2} \| f_n - f \|_W^2 -
- ((a_1(n) - a_1(0)) (F_{xx} - \frac{1}{2} F_{yy}), (f_n - f)_{xx}) - ((a_1(n) - a_1(0)) (F_{yy} - \frac{1}{2} F_{xx}), (f_n - f)_{yy}) -
- 3(F_{xy}(a_1(n) - a_1(0)), (f_n - f)_{xy}) - \frac{E h M_2}{8 L_2^2} \| f_n - f \|_W^2 - \frac{E h}{4} \left\{ (w_{xx}(a_2(n) - a_2(0)), (f_n - f)_{yy}) - (w_{yy}(a_2(n) -
- a_2(0)), (f_n - f)_{xy}) + 2(w_{xy}(a_2(n) - a_2(0)), (f_n - f)_{xy}) \right\} - Z_n,\]

where \( L_1, L_2 > 0 \) are arbitrary numbers and

\[ Z_n = \frac{9}{E h^2} \left\{ B(w_n; f_n, w_n - w) - B(w; f, w_n - w) + B(w_n; f_0, w_n - w) -
- B(w; f_0, w_n - w) \right\} + \frac{E}{2} \left\{ B(w_n; w_n, f_n - f) - B(w_n; w_n, f - f) \right\}.\]

From the compactness of the imbedding \( W^2_2(\Omega) \to W^1_2(\Omega) \) \((n = 2)\) and from (3) we obtain \( \lim_{n \to \infty} Z_n = 0 \). All the members containing the expression \( a_t(n) - a_t(0) \) are estimated in the same way. Let us consider, e.g., the integral

\[ J = (w_{xx}(a_3(n) - a_3(0)), (w_n - w)_{xx}).\]

We have

\[ J = \left( w_{xx} \int_0^t \frac{d}{dt} a_3(D(w + t(w_n - w)); D(F + t(F_n - F))) dt, (w_n - w)_{xx} \right) =
\]

\[ = \left( \sum_{|l| \leq 2} D^l(w_n - w) \int_0^1 \frac{\partial a_3}{\partial \tau_i} w_{xx} d\tau, (w_n - w)_{xx} \right) +
+ \left( \sum_{|l| \leq 2} D^l(f_n - f) \int_0^1 \frac{\partial a_3}{\partial \tau_i} w_{xx} d\tau, (w_n - w)_{xx} \right),\]
where \(i = (i_1, i_2)\) is a multiindex and \(D^iv = \partial^{(i_1)}v/(\partial x^{i_1}\partial y^{i_2})\). Owing to Lemma 3 we conclude from (7) that
\[
\left| \int_0^1 \frac{\partial a_3}{\partial \xi_i} w_{xx} \, dt + \int_0^1 \frac{\partial a_3}{\partial \tau_i} w_{xx} \, dt \right| \leq KC_i \quad \text{for a.e.} \quad (x, y) \in \Omega
\]
and \(|i| \leq 2\). For \(|i| = 2\) we estimate
\[
\left| \int_0^1 \frac{\partial a_3}{\partial \xi_i} w_{xx} \, dt, (w_n - w)_{xx} \right| \leq \delta K \left( \frac{1}{2} \|D^i(w_n - w)\|^2 + \frac{1}{2} \|(w_n - w)_{xx}\|^2 \right)
\]
and
\[
\left| \int_0^1 \frac{\partial a_3}{\partial \xi_i} w_{xx} \, dt, (w_n - w)_{xx} \right| \leq \delta K \left( \frac{1}{2} \|D^i(f_n - f)\|^2 + \frac{1}{2} \|(w_n - w)_{xx}\|^2 \right).
\]
For \(|i| < 2\) we estimate
\[
J_n(1, i) = \left| \int_0^1 \frac{\partial a_3}{\partial \xi_i} w_{xx} \, dt, (w_n - w)_{xx} \right| \leq C_i K \left\| D_i(w_n - w) \right\| \left\| (w_n - w)_{xx} \right\|
\]
and
\[
J_n(2, i) = \left| \int_0^1 \frac{\partial a_3}{\partial \xi_i} w_{xx} \, dt, (w_n - w)_{xx} \right| \leq C_i K \left\| D_i(f_n - f) \right\| \left\| (w_n - w)_{xx} \right\|
\]
Hence and from (16) we obtain
\[
|J| \leq \delta K \left( \|w_n - w\|_W^2 + \|f_n - f\|_W^2 + 3\|(w_n - w)_{xx}\|^2 \right) + G_n(J),
\]
where \(G_n(J) = \sum_{|i| < 2} (J_n(1, i) + J_n(2, i))\) and \(\lim_{n \to \infty} G_n(J) = 0\).

Analogously we estimate, e.g., the integral
\[
I = \left| (F_{yy}(a_1(n) - a_1(0)), (f_n - f)_{yy}) \right| \leq \delta K \left( \|w_n - w\|_W^2 + \|f_n - f\|_W^2 + 3\|(f_n - f)_{yy}\|^2 \right) + G_n(I),
\]
where \(\lim_{m \to \infty} G_n(l) = 0\). Let \(G_n = \sum J_n(J)\), where the sum is taken over all integrals \(J\) corresponding to (15). Summarizing the previous estimates from (14) we conclude that
\[
(17) \quad P_n + Z_n + G_n \geq \|w_n - w\|_W^2 \left( 1 - \frac{3}{2}M_3 - \frac{9L_1^2 M_2}{8Eh} - \frac{Eh M_2 L_2^2}{8} - C\delta \right) + \|f_n - f\|_W^2 \left( \frac{M_0}{2} - \frac{9M_1}{8Eh L_1^2} - \frac{Eh M_2}{8L_2^2} - C\delta \right),
\]
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where $C$ and $\delta$ are from (13) and $\lim_{n \to \infty} G_n = 0$. Let us choose
\[
L_1 = L_2 = \frac{1}{2}(a + (a^2 + 4M_2^2b^2)^{1/2})/b \quad \text{where} \quad a = 1 - \frac{1}{2}M_3 - \frac{1}{2}M_0,
\[
\quad b = 9/(8\epsilon k) + \frac{1}{2}Eh.
\]
If (14) is satisfied then
\[
1 - \frac{1}{2}M_3 - \frac{9L_1^2M_2}{8Eh} - \frac{EhM_3L_2^2}{8} - C\delta > 0
\]
and
\[
\frac{M_0}{2} - \frac{9M_2}{8EhL_1^2} - \frac{EhM_2}{8L_2^2} - C\delta > 0.
\]
Hence and from (17) we conclude that $u \to u$ in $H$ because $\lim_{n \to \infty} G_n = 0$. Thus, Lemma 3 is proved.

Applying known results (see, e.g., [7]) as a consequence of Lemmas 1–4 we have $A(H) = H^*$, i.e., we can formulate the following theorem.

**Theorem 1.** Suppose (5)–(7). If (9) and (14) are fulfilled then there exists a variational solution of $(E_1)$, $(E_2)$, $(B)$ for all $g \in W_2^{-2}$ and $F_0 \in C^2(\Omega)$.

3. **Asymptotical Behaviour of the Solution for $\omega \to 0$**

The system $(E_1)$, $(E_2)$ for $a_i \equiv 1$, $a_i \equiv 0$, $i = 1, 2$ (this is the case we obtain for $\omega \equiv 0$ in the constitutive law) can be identified with the system of von Kármán equations. In this section we shall be concerned with the behaviour of the solutions $u_\omega$ of the operator equations $A_\omega u = G$ for $\omega \to 0$, where $A_\omega \equiv A$ is the operator corresponding to the system $(E_1)$, $(E_2)$. Denote by $A_0 \equiv A_{\omega=0}$ the operator corresponding to the system of von Kármán (i.e. $a_1 = 1$, $a_2 = a_3 = 0$). Evidently, the operator $A_0 : H \to H^*$ is a bounded, continuous and coercive operator with the property $S$. The functions $a_i (i = 1, 2, 3)$ in $(E_1)$, $(E_2)$ need not necessarily be derived from a function $\omega$. Convergence $\omega_n \to 0$ is to be understood in the following sense: $a_{1,n} \rightharpoonup 1$, $a_{i,n} \rightharpoonup 0$ ($i = 1, 2$) on $\Omega \times R^6 \times R^6$.

**Theorem 2.** We assume that the sequences of the functions $(a_i(x, y, \xi, t))_{n=1}^\infty$ ($i = 1, 2, 3$) satisfy (5)–(7) uniformly with respect to $n$ (i.e., the constants $M_i$ ($i = 0, 1, 2, 3$) and $C_j (|j| \leq 2$) are independent of $n$). Suppose (9), (14) and
\[
a_{1,n} \rightharpoonup 1, \quad a_{2,n} \rightharpoonup 0, \quad a_{3,n} \rightharpoonup 0 \quad \text{for} \quad n \to \infty
\]
uniformly on the set $\Omega \times R^6 \times R^6$. Then from each sequence $(u_n)_{n=1}^\infty$ ($u_n \equiv u_{\omega_n}$ is a solution of $A_{\omega_n} u = G$) it is possible to choose a subsequence $(u_{n_k})_{k=1}^\infty$ such that $u_{n_k} \to u$ in $H$, where $u$ is a solution of $A_0 u = G$. 

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Proof. Existence of the solutions \( u_n, n = 1, 2, \ldots \) is guaranteed by Theorem 1. Owing to the assumptions for \( \{a_{ifTJ}\} \) (i = 1, 2, 3) we easily find out that there exists a \( \zeta \in C^2(\Omega) \) with the property (P) and \( C_1, C_2 \) (all independent of \( n \)) such that the estimate

\[
\left< A_{w^k} u, u \right> \geq C_1 \| u \|^2_H - C_2 \quad (C_1 > 0)
\]

holds for all \( u \in H \) (see the proof of Lemma 2). Hence and from \( A_{w^n} u_n = G \) we obtain \( \| u_n \|^2_H \leq C \). Thus there exists a subsequence \( v_k = u_{n_k} \) and \( u \in H \) such that \( v_k \to u \) in \( H \). First we prove \( v_k \to u \) in \( H \) and then \( A_0 u = G \).

For \( D_k = \left< A_k v_k, v_k - u \right> \) we have \( \lim_{k \to \infty} D_k = 0 \) since \( A_k v_k = G \) (\( A_k \equiv A_{w^n} \)). By the same method as in Lemma 4 we obtain

\[
(19) \quad D_k = \left< A_k v_k - A_k u, v_k - u \right> + \left< A_k u, v_k - u \right> \geq C \| v_k - u \|^2_H - \left| \left< A_k u - A_0 u, v_k - u \right> \right| - \left| \left< A_0 u, v_k - u \right> \right|,
\]

where \( C > 0 \) is independent of \( k \) and \( \lim_{k \to \infty} \left| \left< A_0 u, v_k - u \right> \right| = 0 \). Now we estimate

\[
(20) \quad \left| \left< A_k u - A_0 u, v_k - u \right> \right| \leq C_1 \| A_k u - A_0 u \|^2_{H^*} \leq C_1 \| u \|_{H^*} + \| F_0 \|_W \cdot \frac{3}{2} \sup |a_{1,4}(x, y, \xi, \tau)| + \sup \left< a_{1,4}(x, y, \xi, \tau) \right> + \| F_0 \|_W \cdot \frac{3}{2} \sup |a_{3,4}(x, y, \xi, \tau)| \equiv C_1 T_k(\| u \|),
\]

where the supremum is taken over the set \( \Omega \times R^6 \times R^6 \) and \( T_k(\| u \|) \to 0 \) for \( k \to \infty \) because of (18). The last inequality follows easily from (1), (2) and from the definition

\[
\| A_k u - A_0 u \|^2_{H^*} = \sup_{\| v \|_{H^*} \leq 1} \left< A_k u - A_0 u, v \right>,
\]

where \( v = \{ \varphi, \psi \} \in H \). The estimates (20) and (19) imply \( v_k \to u \) in \( H \). Analogously as in (20) we obtain \( \| A_k v_k - A_0 v_k \|^2_{H^*} \leq T_k(\| v_k \|) \) with \( T_k(\| v_k \|) \to 0 \) for \( k \to \infty \) since \( \| v_k \|_H \leq C \). Hence and from the continuity of \( A_0 \) we conclude

\[
G = \lim_{k \to \infty} A_k v_k = \lim_{k \to \infty} A_0 v_k = A_0 u
\]

since \( v_k \to u \) and Theorem 2 is proved.

Consequence of Theorem 2. If there exists a unique solution \( u \) of the system of von Kármán \( A_0 u = G \), then \( u_{\omega_n} \to u \) in \( H \) where \( u_{\omega_n} \) is a solution of \( A_{\omega_n} u = G \).

Now, we prove that the topological degree of \( A \) for small \( \omega \) (i.e., \( |\omega| - 1|, |a_2|, |a_3| \) are sufficiently small) equals that of \( A_0 \). The topological degree for the operators with the property \( S \) was introduced in [7] and is a generalization of the topological degree for continuous mappings in \( E \) with analogous properties (see [7]).

We denote \( G_\omega(v) \equiv \{ w \in H; \| w - v \|_H \leq R, S_\omega(v) \equiv \{ w \in H; \| w - v \|_H = R \} \), \( A_\omega u = Au - g \) and \( A_0,\omega u = A_0 u - g \) (for all \( u \in H \)), where \( g \in H^* \) and \( A \equiv A_\omega \).
Theorem 3. Let (5)-(7), (9) and (14) be satisfied. Suppose $g \in H^*$, $\sup_{\Omega \times R^6 \times R^6} |a_1| < L$, $M_2 < L$, $M_3 < L$. If $L$ is sufficiently small then the topological degree of $A_g$ equals that of $A_{0,g}$ with respect to $R_R(0)$ for sufficiently large $R$, $(R = R(g, L))$.

Proof. From the properties of the operators $A$ and $A_0$ (see Lemmas 1-4) we deduce that the operator

$$A(t, u) = tA_{0,g}u + (1 - t)A_gu$$

defined on $(t, u) \in \langle 0, 1 \rangle \times H$ is continuous (in all the variables) and differs from zero on the set $(0, 1) \times S_R$ for sufficiently large $R = R(g)$. From the $S$-property of $A$ and $A_0$ (see Lemma 4) we easily find out that $t_n \to t \in \langle 0, 1 \rangle$, $u_n \to u$ in $H$ and $\lim_{n \to \infty} A(t_n, u_n) - u = 0$ implies $u_n \to u$ in $H$. Thus, the operators $A_{0,g}$ and $A_g$ are homotopic (see [7]). To prove Theorem 3 it suffices (see [7]) to prove the estimate

$$(21) \quad \|A_gu - A_{0,g}u\|_{H^*} < \|A_{0,g}u\|_{H^*}$$

for all $u \in S_R(0)$. We have

$$(22) \quad \|A_gu - A_{0,g}u\|_{H^*} = \sup_{\|v\| \leq 1} \langle Au - A_0u, v \rangle \leq \frac{3}{2}M_2\|w\|_w +$$

$$+ \frac{9}{4Eh}M_2\|F\|_w + \sup_{\Omega \times R^6 \times R^6} |1 - a_1| \frac{3}{2}\|F\|_w + \frac{Eh}{4}3M_2\|w\|_w,$$

where $u = \{w, f\}$, $F = f + f_0$. On the other hand, the coercivity of $A_0$ yields

$$\|A_0u - g\|_{H^*} \geq \|u\|_{H^*}^{-1} (C_1\|u\|_H^2 - C_2).$$

Hence and from (22) we obtain (21) and Theorem 3 is proved.

Remark. If $u_0$ is an isolated solution of $A_gu = 0$ then the topological degree of $A_g$ with respect to $G_u(r)$ (which is independent of $r$ for sufficiently small $r$) is called the index of $u_0$. Theorem 3 implies the following assertion: If there exist only isolated solutions of the equations

i) $Au - g = 0$, ii) $A_0u - g = 0$

in $G_R(0)$, then the sum of indices of the solutions of i) is equal to the sum of indices of the solutions of ii).
References


Súhrn

O RIEŠENÍ ISTÉHO ZOVŠEOBECNENÉHO SYSTÉMU VON KÁRMÁNOVÝCH ROVNÍC

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